Matrix Algebra: Basic Introduction

1. Matrix is a rectangular array of elements

eg a 3x3 matrx $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

2. Distinguish column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ from row vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ or $\begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$

Often write row vector with a prime and column without

3. Transpose a matrix A signified A' means interchange all rows and colummns
Transpose a vector b signified b' means make a row vector b a column vector.

4. Matrix addition: need to be the same size as add each of the elements to the corresponding ones in the other matrix.

$$A + B \text{ where} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

5. Multiply a matrix by a scalar: multiply all elements

6. Multiply vectors $a' = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$
must be the same size ie of the same order
then

$$a'.b = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \text{a scalar}$$

7. Matrix multiplication
Consider

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A.B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Note that the first matrix needs to have the same number of columns as the second has rows.

So can multiply $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ by $\begin{bmatrix} 10 & 12 \\ 11 & 13 \end{bmatrix}$

Note also that AB is not necessarily equal to BA so need to distinguish pre and post multiplication.

If AB is defined then $(AB)' = B'A'$

8. Unit or identity matrix is a square matrix with ones down the diagonal and zeros elsewhere

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Determinant is an algebraic function of the elements of a matrix which is denoted $|A|$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| = a_{11}a_{22} - a_{12}a_{21}$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then break down into co-factors (2x2) so

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}(-1)^2 + a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}(-1)^3 + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}(-1)^4$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12(-}a_{21}a_{33} + a_{23}a_{31)} + a_{13(}a_{21}a_{32} - a_{22}a_{31})$$

Note:

i. Get appropriate sign by computing determinant and multiplying by $(-1)^{i+j}$

ii. If 2 columns or rows of A are identical then the determinant is identical

iii. The value of a determinant remains the same if you add to any of its rows (columns) any multiple of other rows (columns)

iv. If any row (column) is a linear combination of another row (column) then the determinant is zero.

10. Inverse of a matrix

$$A = A^{-1}$$

$$\Rightarrow A^{-1}A = AA^{-1} = I$$

$$(AB)^{-1} = A^{-1}B^{-1}$$

If $|A| = 0$ the matrix is singular and no inverse exists

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & . & . & . & A_{n1} \\ A_{12} & A_{22} & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ A_{1n} & . & . & . & . & A_{nn} \end{bmatrix}$$

where $A_{ij}$is the co-factor of A (delete the ith row and the jth column and work out the determinant.

Clearly this is a lot of work by hand, but there are many procedures available)

11. Orthogonality: consider vectors a and b then if $a'b = 0$ they are orthogonal
For a matrix orthogonality means

$$A'A = \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = I$$

For an orthogonal matrix the transpose and the inverse are the same, hence $X' = X^{-1}$
also $XX' = I$ and $|X'X| = 1$

12. Rank of a matrix: Given an mxn matrix A:
- the row rank is the maximum number of linearly independent rows in the matrix
- the row rank = the column rank
- $rank(A) \leqslant \min(m,n)$
- $rank(AB)$ is not greater than the $rank(A)$ or the $rank(B)$
- the rank is not altered by pre/post multiplication by a non-singular matrix

13. Trace of a matrix: is the sum of its diagonal elements

$$Trace(A) = Trace(A')$$

$$Trace(A+B) = Trace(A) + Trace(B)$$

$$Trace(AB) = Trace(BA)$$

$$Trace(cA) = cTrace(A)$$

14. Consider vector x of order n and matrix A of order n x n

$$x'Ax = a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{21}x_2x_1 + \ldots + a_{nn}x_n^2$$

$$= \sum_i \sum_j a_{ij}x_ix_j$$

ie quadratic form. (Note $a'x$ would be linear form)

A is positive definite if $x'Ax > 0 \ \forall \ x$
A is positive definite if $x'Ax \geqslant 0 \ \forall \ x$
A is positive definite if $x'Ax < 0 \ \forall \ x$
A is positive definite if $x'Ax \leqslant 0 \ \forall \ x$

15. Differentiation: Can often do operations on vectors/matrices in the same way as scalars

$$\frac{\partial a'x}{\partial x} = a$$

$$\frac{\partial(x'A'x)}{\partial x} = Ax + A'x = 2Ax \text{ if } A \text{ is symmetric}$$

$$\frac{\partial(x'A'y)}{\partial x} = Ay$$

$$\frac{\partial}{\partial A} x'Ax = \begin{bmatrix} x_1^2 & .. & .. & x_1x_n \\ . & . & . & . \\ . & . & . & . \\ x_nx_1 & .. & .. & x_n^2 \end{bmatrix}$$

if symmetric

$$\frac{\partial}{\partial A}(Trace(AB)) = B$$

otherwise

$$\frac{\partial}{\partial A}(Trace(AB)) = B'$$

$$\frac{\partial|A|}{\partial a_{ij}} = A_{ij} \text{ the cofactor of A}$$

$$\frac{\partial|A|}{\partial A} = [A_{ij}] = |A|(A')^{-1}$$

also

$$\frac{\partial \log|A|}{\partial A} = \frac{1}{|A|}\frac{\partial|A|}{\partial A} = (A')^{-1}$$

$$\frac{\partial A^{-1}}{\partial x} = -A^{-1}\frac{\partial A}{\partial x}A^{-1}$$

16. Eigenvalues and eigenvectors, or characteristic roots and vectors
Consider

$$\max \ x'Ax \quad \text{st} \quad x'x = 1$$

Introduce Lagrangian

$$£ = x'Ax - \lambda(x'x - 1)$$

$$\Rightarrow 2Ax - \lambda 2x = 0$$

$$\text{or} \;\; (A - \lambda I)x = 0$$

Now $(A - \lambda I)$ must have rank less than n $\Rightarrow |A - \lambda I| = 0$

Roots of $(A - \lambda I)$ are the characteristic roots of A, the eigenvalues, and the solution vector x is the characteristic vector, or eigenvector

$|A - \lambda I| = 0$ is an nth degree equation in λ and has n roots

For example if A is 3x3 then

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

which is cubic in λ and has 3 roots.

The determinant of any matrix is equal to the product of its eigenvalues

$$D = \begin{bmatrix} \lambda_1 & . & 0 & 0 \\ . & \lambda_2 & . & 0 \\ 0 & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}$$

The trace of a matrix is equal to the sum of its eienvalues

The rank of a matrix is equal to the number of its non zero eigenvalues

If A is positive definite then all its roots are positive definite

If B is positive definite and AB is positive semidefinite then $|A - \lambda B| = 0$ has all roots $\lambda \geqslant 1$; $|A| \geqslant |B|$; and $B^{-1} - A^{-1}$is positive semidefinite.

17. Idempotent matrix is a symmetric matrix A with the property

$$A = A^2$$

if so

$$|A| = 0 \text{ or } 1 \text{ and } Trace(A) = Rank(A)$$

18. Kronecker products: multiply each elemnt of A by B

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

has the property that

$$[A \otimes B]^{-1} = A^{-1} \otimes B^{-1}$$

19. Systems of homogeneous equations

$$\begin{array}{cccccccc} a_{11}x_1 + & a_{12}x_2 + & a_{13}x_3 + & \dots & \dots & +a_{1n}x_n & = & 0 \\ . & . & . & . & . & . & . & . \\ a_{m1}x_1 + & . & . & . & . & +a_{mn}x_n & = & 0 \end{array}$$

$$Ax = 0$$

where x is a column vector and

$$A = \begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ . & . & . & . & . & . \\ a_{m1} & . & . & . & . & a_{mn} \end{array}$$

$\Rightarrow$ x is orthogonal to row vectors of A

Since x is a vector of order n and n is the maximum number of linearly independent vectors of order m

$\Rightarrow$necessary and sufficient considtion for the existence of a non null soultion is $Rank(A) < n$

If $Rank(A) = r$ then there are (n-r) linearly independent vectors that are solutions to this set of equations.

$\Rightarrow$any linear combination of these (n-r) vectors is also a soultion

20. Systems of non homogenous equations: this is when replace the zero by a non zero vector

$$Ax = b$$

so b i linear combination of the columns of A

$\Rightarrow$necessary and sufficient conditions for the existence of a solution to the set of eqautions is

$$Rank(A) = Rank(Ab)$$