Cointegrating VARS

A pth order VAR

- Taking the general VAR

$$y_t = A_0 + \sum_{i=1}^{p} A_i y_{t-i} + \varepsilon_t$$

- This is stable if all the roots of the determinental equation $| \; I - A_1 z - A_2 z^2 - \ldots - A_p z^p \; | = 0$ lie outside the unit circle.
- If we reparameterise the VAR2:

$$y_t = A_0 + A_1 y_{t-1} + A_2 y_{t-2} + \varepsilon_t$$

as:

$$\begin{aligned} y_t - y_{t-1} &= A_0 - (I - A_1 - A_2) y_{t-1} - A_2 (y_{t-1} - y_{t-2}) + \varepsilon_t \\ \Delta y_t &= A_0 - \Pi y_{t-1} + \Gamma \Delta y_{t-1} + \varepsilon_t \end{aligned}$$

- and the VARp as:

$$\Delta y_t = A_0 - \Pi y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + \varepsilon_t .$$

- Which is called the 'cointegrating transformation'
- Notice that this is the vector equivalent of the ADF above for testing for unit roots. Express the Γ_i in terms of the A_i.

Cointegration

- Taking this form of the VAR

$$\Delta y_t = A_0 - \Pi y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + \varepsilon_t$$

- If all the variables, the m elements of y_t, are I(0), Π is a full rank matrix.
- If all the variables are I(1) and not cointegrated, $\Pi = 0$, and a VAR in first differences is appropriate.
- If the variables are I(1) and cointegrated, with r cointegrating vectors, then there are r cointegrating relations, combinations of y_t that are I(0),

  $$z_t = \beta' y_t$$

  where z_t is a $r \times 1$ vector and β' is a $r \times m$ matrix.
- Then we can write the model as:

  $$\Delta y_t = A_0 - \alpha z_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + \varepsilon_t ,$$

  in which the I(0) dependent variable is only explained by I(0) variables and α is a $m \times r$ matrix of 'adjustment coefficients' which measure how the deviations from equilibrium (the r I(0) variables z_{t-1}) feed back on the changes.
- This is called the 'cointegrating transformation'

- It can also be written:

$$\Delta y_t = A_0 - \alpha\beta' y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + \varepsilon_t,$$

  so $\Pi = \alpha\beta'$ has rank $r < m$ if there are r cointegrating vectors.
- If there are $r < m$ cointegrating vectors, then y_t will also be determined by $m - r$ stochastic trends.
- If there is cointegration, some of the α must be non-zero, there must be some feedback on the y_t to keep them from diverging, i.e. there must be some Granger causality in the system.
- If there are r cointegrating vectors and Π has rank r, it will have r non-zero eigenvalues
- Johansen provided a way of estimating the eigenvalues and two tests for determining how many of the eigenvalues are different from zero.
- These allow us to determine r, though the two tests may give different answers.
- The Johansen estimates of the cointegrating vectors β are the associated eigenvectors but there is an 'identification' problem, since the α and β are not uniquely determined.
- We can always choose a non-singular $r \times r$ matrix P such that $(\alpha P)(P^{-1}\beta) = \Pi$ and the new estimates $\alpha^* = (\alpha P)$ and $\beta^* = (P^{-1}\beta)$ would be equivalent, though they might have very different economic interpretations.
- Put differently, if $z_{t-1} = \beta' y_{t-1}$ are I(0) so are $z^*_{t-1} = P^{-1}\beta' y_{t-1}$, since any linear combination of I(0) variables is I(0).
- We need to choose the appropriate P matrix to allow us to interpret the estimates.
- This requires r^2 restrictions, r on each cointegrating vector.
- One of these is provided by normalisation, we set the coefficient of the 'dependent variable' to unity, so if $r = 1$ this is straightforward; but if there is more than one cointegrating vector it requires prior economic assumptions.
- The Johansen identification assumption, that the β are eigenvectors with unit length and orthogonal, do not allow an economic interpretation.
- The EViews identifying assumptions (that the first rxr block of the β matrix is the identity matrix) are only rarely appropriate.
- Microfit allows you to specify the r^2 just identifying restrictions and test any extra 'over-identifying' restrictions.

- As we saw above with the Dickey Fuller regression, there is also a problem with the treatment of the deterministic elements.
- If we have a linear trend in the VAR, and do not restrict the trends, the variables will be determined by $m - r$ quadratic trends.
- To avoid this (economic variables tend to show linear not quadratic trends), we enter the trends in the cointegrating vectors.
- Most programs give you a choice of how you enter trends and intercepts;
- unrestricted intercepts and restricted trends is a good choice for trended economic data.
- Takingthe VAR

$$Z_t = \alpha_0 + \alpha_1 t + \sum_{i=1}^{k} \Phi_i Z_{t-i} + \psi_y w_t + u_t$$

and corresponding cointegrating VAR

$$\Delta y_t = \alpha_{0y} + \alpha_{0x} t - \Pi_y z_{t-1} + \sum_{i=1}^{p-1} \Gamma_{iy} \Delta z_{t-i} + \psi_y w_t + \varepsilon_t$$

$$z_t = (y_t', x_t')'$$

- y_t is an m_y vector of jointly determined (endogenous) $I(1)$ variables
- x_t is an m_x vector of exogenous $I(1)$ variables
- w_t is a $qx1$ vector of exogenous/deterministic $I(0)$ variables
- allow for feedbacks Δy to Δx but not for levels feedbacks
- assumes the xs are not themselves cointegrated

$$\Delta x_t = \alpha_{0x} + \sum_{i=1}^{p-1} \Gamma_{ix} \Delta z_{t-i} + \psi_x w_t + \epsilon_t$$

- So for:

$$\Delta y_t = \alpha_{0y} + \alpha_{1y} t - \Pi_y z_{t-1} + \sum_{i=1}^{p-1} \Gamma_{iy} \Delta z_{t-i} + \psi_y w_t + \varepsilon_t$$

- Options are:
  - $\alpha_{0y} = \alpha_{1y} = 0$ (no intercepts and no trends)
  - $\alpha_{1y} = 0$ and $\alpha_{0y} = \Pi_y \mu_y$ (restricted intercepts and no trends)
    - $\alpha_{0y} = \Pi_y \mu_y$ meaning the intercepts are part of the cointegrating vectors
  - $\alpha_{1y} = 0$ and $\alpha_{0y} \neq 0$(unrestricted intercepts and no trends)
  - $\alpha_{0y} \neq 0$ and $\alpha_{1y} = \Pi_y \gamma_y$ (unrestricted intercepts and restricted trends)
    - $\alpha_{1y} = \Pi_y \gamma_y$ meaning the trends are part of the cointegrating vectors
  - $\alpha_{0y} \neq 0$ and $\alpha_{1y} \neq 0$ (unrestricted intercepts and unrestricted trends)

Example

- Consider a VAR1 in the logarithms of real money and income, which are both I(1) with a linear trend:

$$y_t = a_{10} + a_{11}y_{t-1} + a_{12}m_{t-1} + \gamma_1 t + \varepsilon_{1t}$$
$$m_t = a_{20} + a_{21}y_{t-1} + a_{22}m_{t-1} + \gamma_2 t + \varepsilon_{2t}$$

  and $z_t = m_t - \beta y_t$ is I(0).
- The cointegrating vector is $(1, -\beta)$ and we have normalised the equation by setting the coefficient of m_t to unity. This just identifies the cointegrating vector for r=1.
- The VECM is:

$$\Delta y_t = a_{10} + (a_{11} - 1)y_{t-1} + a_{12}m_{t-1} + \gamma_1 t + \varepsilon_{1t}$$
$$\Delta m_t = a_{20} + a_{21}y_{t-1} + (a_{22} - 1)m_{t-1} + \gamma_2 t + \varepsilon_{2t}.$$

- Imposing the cointegration restriction, it becomes:

$$\Delta y_t = a_{10} - \alpha_1(m_{t-1} - \beta y_{t-1}) + \gamma_1 t + u_{1t}$$
$$\Delta m_t = a_{20} - \alpha_2(m_{t-1} - \beta y_{t-1}) + \gamma_2 t + u_{2t}$$

  thus

$$\Pi = \begin{bmatrix} \alpha_1 & -\alpha_1\beta \\ \alpha_2 & -\alpha_2\beta \end{bmatrix}$$

  which is clearly of rank 1, since a multiple of the first column equals the second column.
- A natural over-identifying restriction to test in this context would be that $\beta = 1$.
- To restrict the trend we could put it in the cointegrating vector, saving one parameter:

$$\Delta y_t = a_{10} - \alpha_1(m_{t-1} - \beta y_{t-1} + \gamma t) + u_{1t}$$
$$\Delta m_t = a_{10} - \alpha_2(m_{t-1} - \beta y_{t-1} + \gamma t) + u_{2t}$$