<u>Introduction to Matrix Algebra</u>

1. A Matrix is a rectangular array of elements, e.g. a 3x3 matrix:

A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. An mxn matrix has m rows and n columns.

2. A column vector is of the form e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, a row vector is of the form e.g. [2 -3 4].

3. The transpose of matrix A, denoted A', is obtained by interchanging the rows and columns of A. The transpose of a column vector is the same vector shown as a row, and vice versa. E.g. if A = $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ then A' = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

4. We can add matrices (or vectors) if they are of the same size, by simply adding the corresponding elements in each position of the matrix. So if A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then A+B = $\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$.

5. We can multiply a matrix or vector by a scalar (number) λ by multiplying every element of the matrix or vector by λ. E.g. $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$.

6. We can take the scalar product (or dot product) of two vectors of the same size by multiplying corresponding elements and adding. Thus

$$[a_1\ a_2\ a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

7. We can multiply matrices as follows: if A is an mxn matrix and B is an nxif A is an mxn matrix and B is an nxk matrix, then AB is a matrix, then AB is an mxk matrix, with AB_{ij} = (i'th row of A).(j'th row of B). Thus if A = $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 1 \\ -2 & 2 \\ 1 & 5 \end{bmatrix}$,

then AB = $\begin{bmatrix} 2*4+1*(-2)+3*1 & 2*1+1*2+3*5 \\ 1*4+0*-2+4*1 & 1*1+0*2+4*5 \end{bmatrix} = \begin{bmatrix} 9 & 19 \\ 8 & 21 \end{bmatrix}$.

- Note that A must have as many columns as B has rows.

- Note also that in general AB != BA, so we need to distinguish pre- and post-multiplication.
- However, (AB)C = A(BC) – matrix multiplication is associative.

- If AB is defined then (AB)' = B'A'.

8. The Identity or unit matrix is a square matrix with ones down the diagonal and zeroes elsewhere. I_n is the identity nxn matrix. E.g. I_3 = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

I_n has the property that for any other matrix A, I_n.A = A.I_n = A, where these products exist. (So I leaves other matrices constant under multiplication).

9. The Determinant of a square matrix is an algebraic function of the elements of a matrix A, denoted |A|. It is calculated as follows:

If A is 2x2, A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then |A| = ad-bc.

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then we calculate |A| by multiplying each element of the top row by the corresponding co-factor: a 2x2 matrix obtained by deleting the row and column we're currently looking at, multiplied by an appropriate sign, as follows:

|A| = a_{11}* $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ - a_{12}* $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ + a_{13}* $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

In fact, <u>we can calculate the determinant by expanding by any row or column</u>. (Multiplying each element of a row or column by the corresponding cofactor).

The sign by which we multiply each 2x2 submatrix to obtain the co-factor is given by the following:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

For larger matrices, we again choose a row or column to expand by, and multiply each element by the corresponding cofactor.

Notes:

- The sign can be computed as $(-1)^{i+j}$ for the co-factor corresponding to the $(I,j)^{th}$ position.
- If two columns or rows of A are identical then $|A|=0$.
- More generally, if any row or column is a linear combination of the other rows or columns, then the determinant is 0.
- $|I_n| = 1$.
- $|AB| = |A|*|B|$.
- $|A'| = |A|$.

10. The <u>inverse</u> of an nxn matrix A, denoted A^{-1}, if it exists, is an nxn matrix such that $A*A^{-1} = A^{-1}*A = I_n$.

- Note that since $|A|*|A^{-1}| = |A*A^{-1}| = |I| = 1$, therefore $|A^{-1}| = 1/|A|$.
- Hence, if $|A| = 0$, then A^{-1} cannot exist. We say A is <u>singular</u>.
- Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. (Same other way round).
- Hence, $B^{-1}A^{-1}$ is the inverse of AB, in other words $(AB)^{-1} = B^{-1}A^{-1}$.

If the inverse exists, it is calculated as follows:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \ldots\ldots & & A_{n1} \\ A_{12} & A_{22} & & & . \\ . & . & . & & . \\ . & . & & . & . \\ A_{1n} & A_{2n} & \ldots. & & A_{nn} \end{bmatrix}$$

Where A_{ij} is the ij'th *co-factor,* i.e. the (n-1)x(n-1) sub-determinant obtained by deleting the i'th row and j'th column, multiplied by the appropriate sign. Note that we put A_{ij} in the (j,i)'th position of A^{-1}, so we put each co-factor in the transpose of its position.

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (a single element co-factor is just the value of the element, times the appropriate sign).

(3x3 example on whiteboard….)

There are various manual and computer procedures to simplify calculating determinants and inverses.

11. Two (column) vectors **a** and **b** of the same size are <u>orthogonal</u> if **a'b** = **0** (the zero vector).

A matrix A is said to be Orthogonal if AA' = A'A = I. For an orthogonal matrix, A' = A^{-1}. Hence, |A'| = 1/|A|, but since |A'| = |A|, it follows that |A| = 1 or -1.

12. <u>Linear independence and dependence, and dimensions.</u>

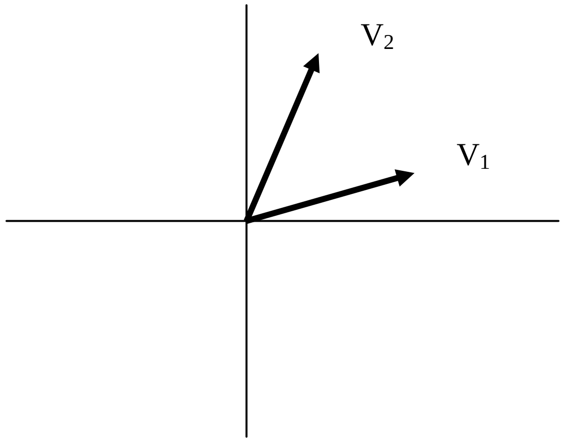
A set of vectors $V_1, V_2, \ldots, V_n$ of the same size are said to be <u>linearly dependent</u> if some non-zero <u>linear combination</u> of the vectors is equal to the zero vector. That is, if there exist scalars $\lambda_1,\ldots, \lambda_n$, not all zero, such that

$\lambda_1 V_1 + \lambda_2 V_2 + \ldots + \lambda_n V_n = 0.$

If no such set of scalars exist (other than $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$), then we say $V_1, \ldots V_n$ are <u>linearly independent</u>.

- Another way of saying this is that if a set of vectors is linearly independent, it is not possible to construct any one of them as a linear combination of the others – each one conveys 'new information'.

- We call the set of all possible linear combinations of a set of vectors $V_1,\ldots,V_n$ the <u>span</u> of that set of vectors, that is Span($V_1,\ldots V_n$) = $\{\lambda_1 V_1 + \ldots + \lambda_n V_n |$ all possible values of $\lambda_1,\ldots,\lambda_n\}$.
- If we have n linearly independent vectors, then their span is an <u>n-dimensional</u> space.
- If some vectors in a set are linear combinations of the others, then they add nothing to the span – we can delete them until we have a linearly independent set.



- V_1 and V_2 are linearly independent – they span a 2-dimensional space.
- If we were to add a third vector in the plane, it would be a linear combination of the other two, and add nothing to the span. (The set would be linearly dependent.)
- If we added a third vector V_3 that came out of the page, this would clearly not be a linear combination of the other two. Then $\{V_1,V_2,V_3\}$ would be a linearly independent set, spanning a 3-dimensional space.

13. Rank of a matrix

Given an mxn matrix A:

- The row rank is the maximum number of linearly independent rows in the matrix
- The column rank is the maximum number of linearly independent columns
- It can be shown row rank = column rank, so we can just talk about the rank of a matrix.
- That is, the rank of the matrix is the dimension of the space spanned by its rows or columns.
- Rank(A) <= min(m,n).
- Rank(AB) <= min(Rank(A),rank(B)).
- If A is non-singular (A^{-1} exists), then rank(AB)=rank(BA)=rank(A)
- Recall: A^{-1} exists if and only if |A| <> 0; and |A| = 0 if and only if some row or column is a linear combination of the others; in other words, A^{-1} exists if and only if the rows (columns) of A are linearly independent. If A is an nxn matrix, this means its rank is also equal to n. Therefore we sometimes say that a non-singular matrix has full rank.

14. The trace of a square matrix is equal to the sum of its diagonal elements.

- Trace(A) = Trace(A')
- Trace(A+B) = Trace(A)+Trace(B)
- Trace(AB) = Trace(BA)
- Trace(cA) = cTrace(A) where c is a scalar.

15. Quadratic forms

Consider a vector *x* of order (size) n and an nxn matrix A. Then

$$x'Ax = a_{11}x_1^{\ 2} + a_{12}x_1x_2 + \ldots + a_{1n}x_1x_n + a_{21}x_2x_1 + a_{22}x_2^{\ 2} + \ldots + a_{nn}x_n^{\ 2}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \quad .$$

We call this function of ($x_1,\ldots,x_n$) the quadratic form corresponding to the matrix A.

For example, if A = $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ then the quadratic form is given by $2x_1^{\ 2} + 2x_1x_2 + 3x_2^{\ 2}$.

(We get the $2x_1x_2$) from adding up the off-diagonal elements.)

We say A is positive definite if x'Ax > 0 for all vectors *x*.

A is positive semi-definite if x'Ax >= 0 for all x

A is negative definite if x'Ax < 0 for all x

A is negative semi-definite if x'Ax <=0 for all x.