

1 Matrix Algebra: Basic Introduction

1. Matrix is a rectangular array of elements

eg a 3x3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

2. Distinguish column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ from row vector $[1 \ 2 \ 3]$ or $[1, \ 2, \ 3]$

Often write row vector with a prime and column without

3. Transpose a matrix A signified A' means interchange all rows and columns
Transpose a vector b signified b' means make a row vector b a column vector.

4. Matrix addition: need to be the same size as add each of the elements to the corresponding ones in the other matrix.

$$A + B \text{ where } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

5. Multiply a matrix by a scalar: multiply all elements

6. Multiply vectors $a = [a_1 \ a_2 \ a_3]$ and $b = [b_1 \ b_2 \ b_3]$
must be the same size ie of the same order

then

$$a' \cdot b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \ b_2 \ b_3] = a_1 b_1 + a_2 b_2 + a_3 b_3 = \text{a scalar}$$

7. Matrix multiplication

Consider

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Note that the first matrix needs to have the same number of columns as the second has rows.

So can multiply $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ by $\begin{bmatrix} 10 & 12 \\ 11 & 13 \end{bmatrix}$

Note also that AB is not necessarily equal to BA so need to distinguish pre and post multiplication.

If AB is defined then $(AB)' = B'A'$

8. Unit or identity matrix is a square matrix with ones down the diagonal and zeros elsewhere

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Determinant is an algebraic function of the elements of a matrix which is denoted $|A|$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| = a_{11}a_{22} - a_{12}a_{21}$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then break down into co-factors (2x2) so

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} (-1)^2 + a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} (-1)^3 + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} (-1)^4 \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Note:

- i. Get appropriate sign by computing determinant and multiplying by $(-1)^{i+j}$
- ii. If 2 columns or rows of A are identical then the determinant is identical
- iii. The value of a determinant remains the same if you add to any of its rows (columns) any multiple of other rows (columns)
- iv. If any row (column) is a linear combination of another row (column) then the determinant is zero.

10. Inverse of a matrix

$$A = A^{-1}$$

$$\implies A^{-1}A = AA^{-1} = I$$

$$(AB)^{-1} = A^{-1}B^{-1}$$

If $|A| = 0$ the matrix is singular and no inverse exists

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & \cdot & \cdot & \cdot & A_{n1} \\ A_{12} & A_{22} & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ A_{1n} & \cdot & \cdot & \cdot & \cdot & A_{nn} \end{bmatrix}$$

where A_{ij} is the co-factor of A (delete the i^{th} row and the j^{th} column and work out the determinant).

Clearly this is a lot of work by hand, but there are many procedures available)

11. Orthogonality: consider vectors a and b then if $a'b = 0$ they are orthogonal

For a matrix orthogonality means

$$A'A = \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = I$$

For an orthogonal matrix the transpose and the inverse are the same, hence $X' = X^{-1}$

also $XX' = I$ and $|X'X| = 1$

12. Rank of a matrix: Given an $m \times n$ matrix A :

the row rank is the maximum number of linearly independent rows in the matrix

the row rank = the column rank

$rank(A) \leq \min(m, n)$

$rank(AB)$ is not greater than the $rank(A)$ or the $rank(B)$

the rank is not altered by pre/post multiplication by a non-singular matrix

13. Trace of a matrix: is the sum of its diagonal elements

$$Trace(A) = Trace(A')$$

$$Trace(A + B) = Trace(A) + Trace(B)$$

$$Trace(AB) = Trace(BA)$$

$$Trace(cA) = cTrace(A)$$

14. Consider vector x of order n and matrix A of order $n \times n$

$$\begin{aligned} x'Ax &= a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{21}x_2x_1 + \dots + a_{nn}x_n^2 \\ &= \sum_i \sum_j a_{ij}x_ix_j \end{aligned}$$

ie quadratic form. (Note $a'x$ would be linear form)

A is positive definite if $x'Ax > 0 \forall x$

A is positive definite if $x'Ax \geq 0 \forall x$
 A is positive definite if $x'Ax < 0 \forall x$
 A is positive definite if $x'Ax \leq 0 \forall x$

15. Differentiation: Can often do operations on vectors/matrices in the same way as scalars

$$\frac{\partial a'x}{\partial x} = a$$

$$\frac{\partial(x'A'x)}{\partial x} = Ax + A'x = 2Ax \text{ if } A \text{ is symmetric}$$

$$\frac{\partial(x'A'y)}{\partial x} = Ay$$

$$\frac{\partial}{\partial A} x'Ax = \begin{bmatrix} x_1^2 & \dots & \dots & x_1x_n \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ x_nx_1 & \dots & \dots & x_n^2 \end{bmatrix}$$

if symmetric

$$\frac{\partial}{\partial A} (\text{Trace}(AB)) = B$$

otherwise

$$\frac{\partial}{\partial A} (\text{Trace}(AB)) = B'$$

$$\frac{\partial |A|}{\partial a_{ij}} = A_{ij} \text{ the cofactor of } A$$

$$\frac{\partial |A|}{\partial A} = [A_{ij}] = |A| (A')^{-1}$$

also

$$\frac{\partial \log |A|}{\partial A} = \frac{1}{|A|} \frac{\partial |A|}{\partial A} = (A')^{-1}$$

$$\frac{\partial A^{-1}}{\partial x} = -A^{-1} \frac{\partial A}{\partial x} A^{-1}$$

16. Eigenvalues and eigenvectors, or characteristic roots and vectors
 Consider

$$\max x'Ax \text{ st } x'x = 1$$

Introduce Lagrangian

$$\mathcal{L} = x'Ax - \lambda(x'x - 1)$$

$$\implies 2Ax - \lambda 2x = 0$$

$$\text{or } (A - \lambda I)x = 0$$

Now $(A - \lambda I)$ must have rank less than $n \implies |A - \lambda I| = 0$

Roots of $(A - \lambda I)$ are the characteristic roots of A , the eigenvalues, and the solution vector x is the characteristic vector, or eigenvector

$|A - \lambda I| = 0$ is an n^{th} degree equation in λ and has n roots

For example if A is 3×3 then

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

which is cubic in λ and has 3 roots.

The determinant of any matrix is equal to the product of its eigenvalues

$$D = \begin{bmatrix} \lambda_1 & \cdot & 0 & 0 \\ \cdot & \lambda_2 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{bmatrix}$$

The trace of a matrix is equal to the sum of its eigenvalues

The rank of a matrix is equal to the number of its non zero eigenvalues

If A is positive definite then all its roots are positive definite

If B is positive definite and AB is positive semidefinite then $|A - \lambda B| = 0$ has all roots $\lambda \geq 1$; $|A| \geq |B|$; and $B^{-1} - A^{-1}$ is positive semidefinite.

17. Idempotent matrix is a symmetric matrix A with the property

$$A = A^2$$

if so

$$|A| = 0 \text{ or } 1 \text{ and } \text{Trace}(A) = \text{Rank}(A)$$

18. Kronecker products: multiply each element of A by B

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

has the property that

$$[A \otimes B]^{-1} = A^{-1} \otimes B^{-1}$$

19. Systems of homogeneous equations

$$\begin{array}{cccccc} a_{11}x_1 + & a_{12}x_2 + & a_{13}x_3 + & \dots & \dots & + a_{1n}x_n & = & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1}x_1 + & \cdot & \cdot & \cdot & \cdot & + a_{mn}x_n & = & 0 \end{array}$$

$$Ax = 0$$

where x is a column vector and

$$A = \begin{matrix} & a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & & \cdot & \cdot & \cdot & \cdot & a_{mn} \end{matrix}$$

$\implies x$ is orthogonal to row vectors of A

Since x is a vector of order n and n is the maximum number of linearly independent vectors of order n

\implies necessary and sufficient condition for the existence of a non null solution is $\text{Rank}(A) < n$

If $\text{Rank}(A) = r$ then there are $(n-r)$ linearly independent vectors that are solutions to this set of equations.

\implies any linear combination of these $(n-r)$ vectors is also a solution

20. Systems of non homogeneous equations: this is when replace the zero by a non zero vector

$$Ax = b$$

so b is linear combination of the columns of A

\implies necessary and sufficient conditions for the existence of a solution to the set of equations is

$$\text{Rank}(A) = \text{Rank}(Ab)$$