Some Notes on Consumer Theory

1. Introduction

In this lecture we examine the theory of duality in the context of consumer theory and its use in the measurement of the benefits of price and other changes. Duality is not a topic that is often not covered in undergraduate microeconomics. Yet it is a powerful tool in the analysis of consumer behaviour. It enables us to derive theoretical results more easily. Much — if not most — applied demand analysis nowadays uses duality. As indicated above, it is also used in the measurement of benefits in cost-benefit analysis. Finally, much of the material we cover on duality in the context of the consumer has relevance to the firm. The first part of the lecture outlines the basics of duality theory. The second shows how it is used in the measurement of price changes.

2. Duality in consumer theory

2.1. <u>The primal problem and Marshallian demands</u>

In undergraduate microeconomics, consumers are typically viewed as choosing a consumption bundle, x, so as to maximise utility, $u(x)$, subject to their budget constraint, $px=M$. Thus the problem is

(1) Choose x so as to maximise $u(x)$ subject to $px=M$.

To solve this we set up the Lagrangian function

(2) $$L = u(x) + \lambda\left[M - px\right],$$

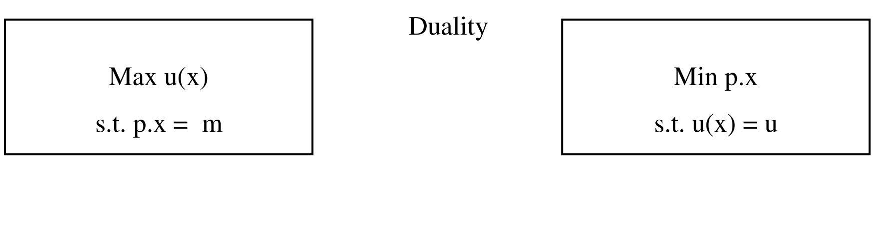
differentiate it with respect to each of the x_i and λ, thereby obtaining the familiar conditions

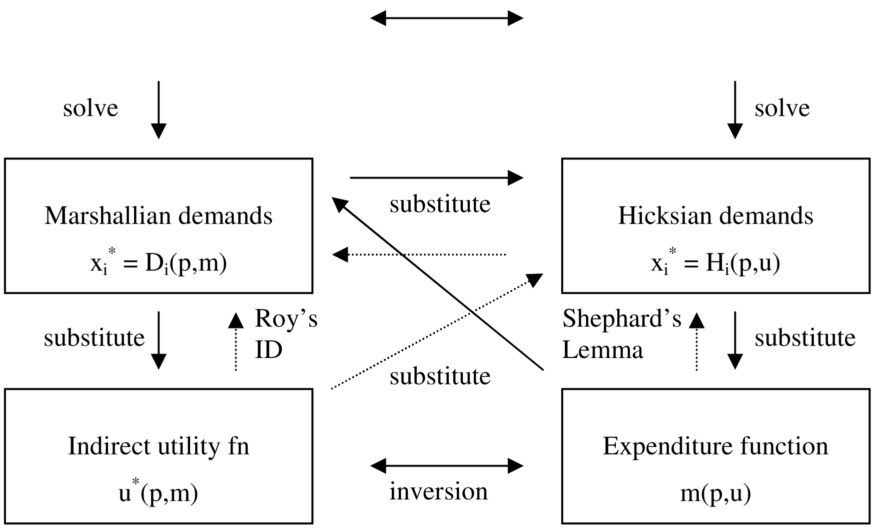
(3) $$p_i / p_j = u_i / u_j ,$$

which require that the marginal rate of substitution between good i and good j (the slope of the indifference curve) equals the price ratio of the these two goods (the slope of the budget line). We thus derive an optimal consumption bundle, x^*, whose value depends on the price vector, p, and the available income, M. We can therefore talk of a demand function for good i of the form

(4) $$x_i^* = D_i\left(p, M\right),$$

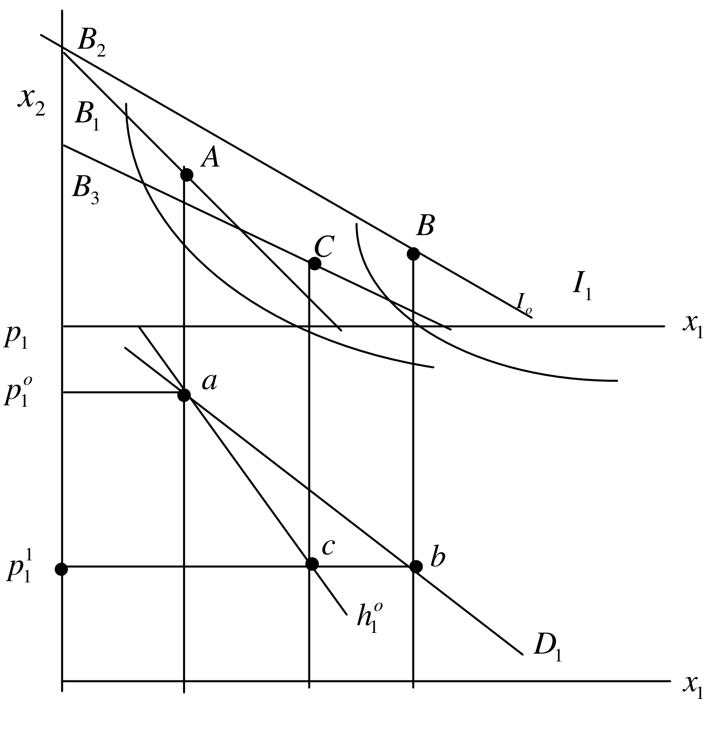
which is known as the *Marshallian* demand function. This gives us a start in the map shown in Fig 1.





The derivation in the two-good case of the Marshallian demand function is shown in Fig 2. The consumer starts on budget line B_1 at point A, reaching indifference curve I_0. The price of x_1 then falls from p_1^0 to p_1^1, giving a new budget line B_2 and a new optimum at B on indifference curve I_1. In the lower part of Fig 2, points a and b correspond to points A and B in the upper part of Fig 1. The Marshallian demand for good 1 curve passes through points a and b and is labelled D_1. Changes in p_1 thus result in movements *along* D_1.

Fig 2



Recall the location of D_1 depends on the prices of the other goods (in this case just p_2) and income, M. For example, if income rises, and x_1 is a normal good, the new optimum will be on I_1 somewhere to the left of B but to the right of A. There will be a new point in the lower part of Fig 1 corresponding to D, labelled d, that lies above D_1. Through point d there will be Marshallian demand curve for x_1 corresponding to the new higher income level. Note, though, that if income were to, say, double, *and* both prices were to double, the demand for x_1 (and x_2) wouldn't change. We say that the *Marshallian demands are homogeneous of degree zero in prices and income together*. More formally, for any scalar $\theta>0$ it is the case that $D_i(\theta p,\theta M)=D_i(p,M)$.

2.2. The dual problem and Hicksian demands

The problem above is known as the *primal problem*. An alternative approach — known as the *dual* of the problem above — is to view consumers as choosing a consumption bundle, x, so as to minimise the expenditure, px, required to attain a specific level of utility, u. Thus the dual problem to (1) is

(5) Choose x so as to minimise $M=px$ subject to $u(x)=u$.

This is shown in the map in Fig 1. The relevant Lagrangian for the dual problem is

(6) $$L=-px+\mu\left[u(x)-u\right].$$

This gives conditions

(7) $$p_i/p_j=u_i/u_j,$$

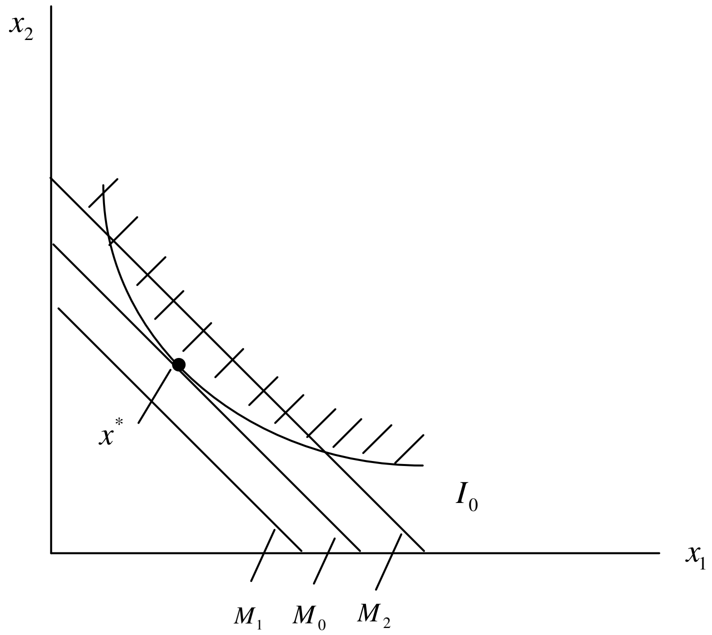
which are identical to those obtained from the primal problem.

This is illustrated in Fig 3. In the original problem, the consumer moves along the budget line until the highest attainable indifference curve is reached. In the dual problem, she moves along the indifference curve until the lowest iso-expenditure line is reached. (Recall that lower iso-expenditure lines are associated with lower expenditure levels.) Solving the dual problem will produce an optimal consumption bundle, whose value will depend on the price vector, p, as in the original problem, but also on the target level of utility chosen, u. If, as our target level of utility, we choose the level of utility attained in the original problem, then it is clear from Fig 3 that the consumption bundle chosen in the dual problem must be the same as that chosen in the original problem, i.e. x^*. But, of course, the arguments of the demand function are now p and u, rather than p and M. This gives a demand function associated with the dual problem of the form

(8) $$x_i^*=H_i(p,u),$$

which is known as the *Hicksian demand*. The route from the dual problem to the Hicksian demand functions is shown in the map in Fig 1.

Fig 3



Sometimes Hicksian demand functions are also known as *compensated* demand functions, because the consumer's utility is held constant when prices change and hence the consumer is "compensated" for them. This is shown in Fig 2. We reduce p_1, as before, but instead of holding *M* constant, as we did when deriving the Marshallian demand function, we hold *utility* constant — i.e. we keep the individual on indifference curve I_0. This gives us budget line B_3 and produces the optimal bundle shown as *C*. This lies to the right of *A* (given convexity of I_0) but to the left of *B* (assuming x_1 is normal). The move from *A* to *C* is, of course, the *substitution effect*.

We can trace out the Hicksian demand function in the lower part of Fig 2. Point *c* corresponds to point *C* in the upper part of the diagram. We know that points *a* and *c* lie on the Hicksian demand curve corresponding to the utility level associated with I_0. This Hicksian demand curve is labelled h_1^0. The slope of the Hicksian demand curve, $\partial H_i / \partial p_i$ is the equal to the substitution effect. We'll consider later the properties of the Hicksian demand function.

2.3. The expenditure function

Substituting the Hicksian demand functions into the objective function of the dual problem, *px*, gives

(9) $$\sum_i p_i x_i^* = \sum_i p_i H_i(p,u) = m(p,u).$$

The function $m(p,u)$ is known as the *expenditure function* or *cost function*. It shows the minimum expenditure required to achieve a given level of utility, conditional on a particular price vector. This role of the expenditure function is shown in the map in Fig 1.

There are various important properties of the expenditure function we need to know. The first is the following:

P1: *$m(p,u)$ is homogeneous of degree one in prices.*

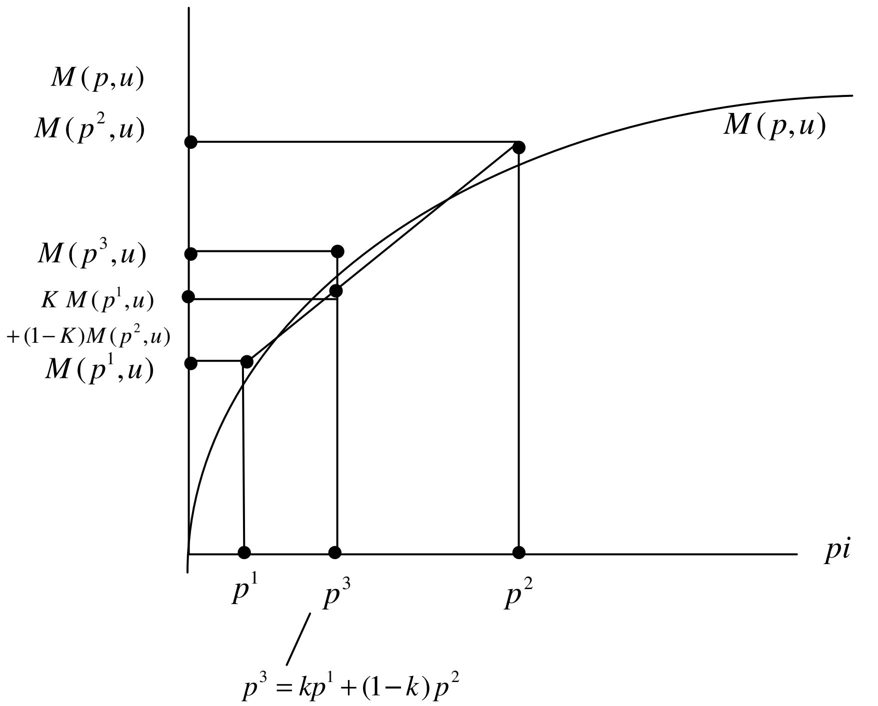
This means that if all prices double, expenditure doubles. This result is easily shown. Fix u at u' and p at p', and denote the corresponding optimal bundle by x'. Then the corresponding expenditure function is $m'=m(p',u')$. Pick a new price vector kp', where $k>0$. Relative prices don't change, so if we hold u constant at u', it must be the case that x' is still the optimal bundle. The new level of expenditure is therefore equal to $m(kp',u)=kp'x'=km'$.

Property P1 tells what happens when all prices change by the same proportion and we hold utility constant. The next property tells us what happens when just *one* price changes and we hold utility constant. The property is:

P2. *The expenditure function is concave in prices.*

This is shown in Fig 4. If $m(p,u)$ is strictly concave, as shown, the property means that if we double p_1, expenditure *less than doubles*. Convexity of indifference curves guarantees strict concavity, but convexity isn't required for non-strict convexity.

Fig 4



We can prove concavity as follows. Suppose we have two price vectors, p^1 and p^2, and suppose that when utility is fixed at u^0, the optimal bundles corresponding to these vectors are x^1 and x^2 — see Fig 5. We have the corresponding expenditures, $m(p^1,u^0)$ and $m(p^2,u^0)$. Next take a linear combination (i.e. a weighted average) of the price vectors,

(10) $p^3 = kp^1 + (1\text{-}k)p^2$,

and let x^3 be the optimal bundle when prices are p^3 and utility is u^0 — see Fig 5. Concavity means that the cost of reaching utility level u^0 at price vector p^3 must not be less than the weighted average of $m(p^1,u^0)$ and $m(p^2,u^0)$ — see Fig 4. Thus concavity means

(11) $m\left(p^3,u^0\right) \geq km\left(p^1,u^0\right) + (1-k)m\left(p^2,u^0\right)$.

By the axiom of cost-minimisation, this must indeed be true. The LHS of the inequality in (11) is equal to:

(12) $m\left(p^3,u^0\right) = p^3x^3 = \left[kp^1 + (1-k)p^2\right]x^3 = kp^1x^3 + (1-k)p^2x^3$

Consider the terms p^1x^3 and p^2x^3. These indicate the cost of bundle x^3 at prices p^1 and p^2 respectively. We know that bundles x^1, x^2 and x^3 are all on the same indifference curve corresponding to utility level u^0. We also know that x^1 minimises the cost of reaching u^0 when prices are p^1, so we can be sure that the cost of buying the bundle x^3 when prices are p^1 cannot be less than the cost of buying x^1 at the same price vector — cf. Fig 5. In other words,

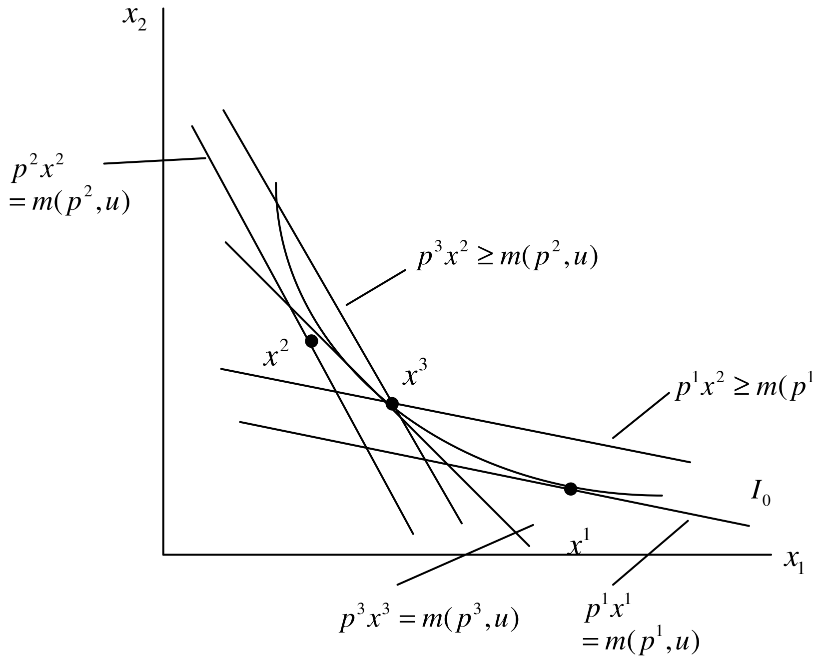
(13) $p^1x^1 = m\left(p^1,u^0\right) \leq p^1x^3$.

Indeed, if indifference curves are convex, as in Fig 5, (15) will be a strict equality. By the same reasoning, we can write:

(14) $p^2x^2 = m\left(p^2,u^0\right) \leq p^2x^3$.

Combining (13) and (14) with (12) makes it clear that (11) must be true.

Fig 5



The third property we need to know is *Shephard's Lemma*. This says that the partial derivatives of the expenditure function with respect to prices are the Hicksian demand functions. This allows us to work back in Fig 1 from the expenditure function to the cost-minimising demands that underlie it — see Fig 1. Shephard's Lemma stated formally is:

P3. *Shephard's Lemma*: $\partial m / \partial p_i = x_i^* = H_i(p,u)$.

To prove this, consider an arbitrary price vector p^0, a utility level u^0, and the corresponding vector of optimal choices $x^0=H(p^0,u^0)$. Associated with this is the cost function $m(p^0,u^0)$. If we graph this cost function, we get Fig 6. Suppose we vary p_1 but keep all other prices unchanged, and we keep the bundle unchanged at x_0. Then, by varying p_1, we trace out the straight line:

(15) $$Z = p_1 H_1(p^0,u^0) + \sum_{j=2}^{n} p_j^0 H_j(p^0,u^0),$$

which is the line in Fig 7. Obviously at p_1^0, we have

(16) $$Z^0 = p_1^0 H_1(p^0,u^0) + \sum_{j=2}^{n} p_j^0 H_j(p^0,u^0) = m(p^0,u^0),$$

so that at p_1^0, $m(p^0,u^0)$ and Z are tangential, so that the slope of the cost function in Fig 7 is equal to the slope of Z, which from eqn (15) is clearly just x_1^0. This proves the result and helps see the intuition behind Shephard's Lemma. Suppose you buy 100 units of x_1 a week. The price of x_1 then rises by £1. As a first approximation, your expenditure would have to rise by £100 to allow you maintain the same utility. Fig 5 shows why this is true only as a first approximation — it is only true for infinitesimally small price changes, since when the price of x_1 rises, you will substitute away from x_1 (see Fig 7).

Fig 6

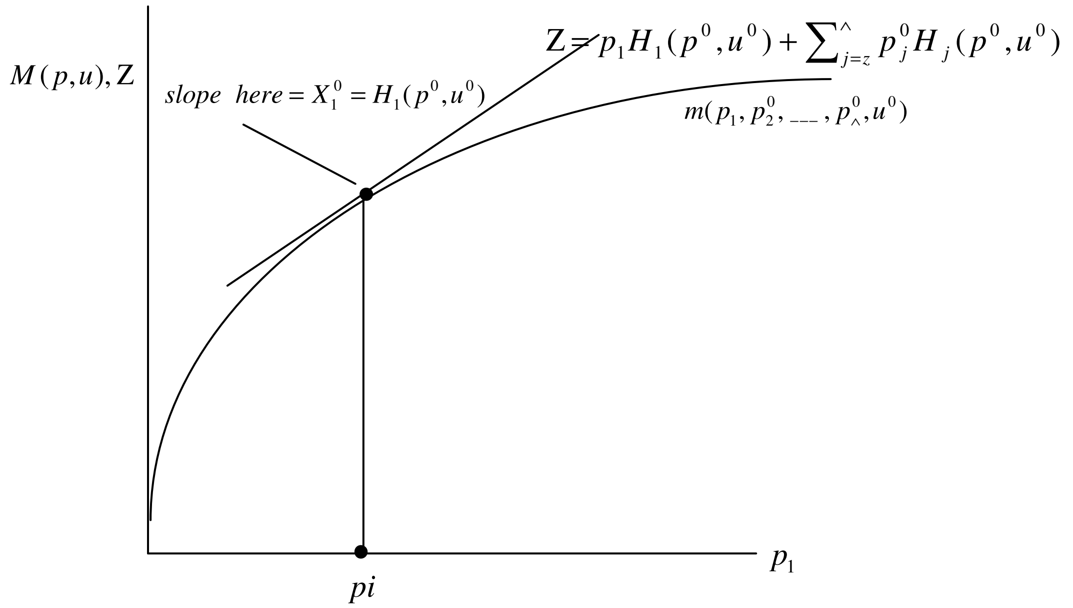
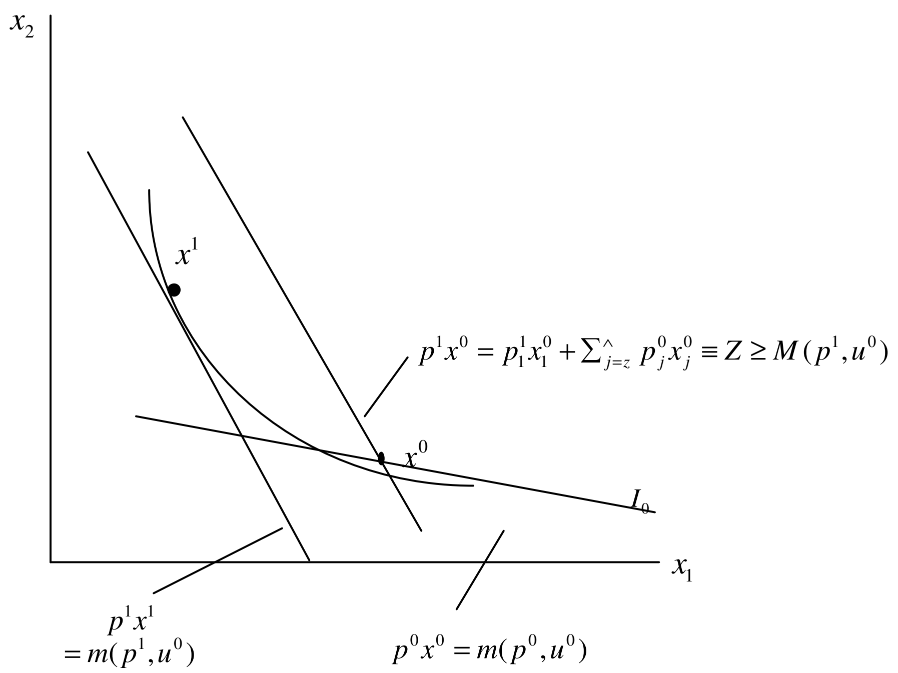


Fig 7



Properties P1 and P3 together have an important implication: *Hicksian demands are homogeneous of degree zero in prices*. Thus doubling prices, with utility held constant, leaves the demand for good *i* unchanged. That this must be the case ought to be clear from Fig 3.*

2.4. The indirect utility function and Roy's identity

We're ready now to introduce another concept — the *indirect utility function*. This is derived not from the dual problem but from the original problem (1). To get the indirect utility function, we simply substitute the Marshallian demands, $x_i^* = D_i(p, M)$, into the utility function, $u(x)$, to get

$$(17) \qquad u(x_1^*,\ldots,x_n^*) = u(D_1(p,M),\ldots,D_n(p,M)) = u^*(p,M).$$

The function $u^*(p,M)$ shows *maximised* utility as a function of the ultimate determinants of utility — prices and income. The derivation of the indirect utility function from the Marshallian demands is shown in Fig 1.

The indirect utility function is related to the expenditure function. Since $m(p,u)=M$, we can rearrange or "invert" $m(p,u)$ to give u as a function of p and M. This will give us $u=u^*(p,M)$. The converse also applies — we can invert the indirect utility function $u=u^*(p,M)$ to get the expenditure function, i.e. $M=m(p,u)$. The link between the expenditure function and the indirect utility function is also shown in Fig 1.

There are other links between the various functions in Fig 1 worth mentioning. We might wish to generate Marshallian demands from the expenditure function. This we can do by substituting the indirect utility function into the Hicksian demands:

$$(18) \qquad x_i^* = h_i(p,u) = h_i(p,u^*(p,M)) = D_i(p,M).$$

The whole thing can also be done in reverse — starting with the Marshallian demands, we can substitute the expenditure function into the Marshallian demands to get the Hicksian demands:

$$(19) \qquad x_i = D_i(p,M) = D_i(p,m(p,u)) = h_i(p,u).$$

These links are also shown in Fig 1.

One last link is useful. We can work back from the indirect utility function to the Marshallian demands. Since the expenditure function is the inverse of the indirect utility function, we can write

$$(20) \qquad u^*(p,m(p,u)) \equiv u.$$

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* It can be shown mathematically. P4 tells us that the Hicksian demand for good *i* is the derivative of the cost function with respect to the price of good *i*. P1 tells us that the cost function is homogeneous of degree one. It is the case that the derivative of a function that is homogeneous of degree *n* is itself homogeneous of degree *n*-1. So, it follows that Hicksian demands are homogenous of degree zero.

Differentiating eqn (20) with respect to p_i, allowing m to vary so as to hold u constant, gives

(21) $$\frac{\partial u^*}{\partial p_i}+\frac{\partial u^*}{\partial M}\frac{\partial m}{\partial p_i}=0.$$

By Shephard's Lemma, we have $\partial m/\partial p_i = H_i(p,u)$. But it is also true that

(22) $$x_i^* = H_i(p,u) = D_i(p,M),$$

i.e. the amount demanded in equilibrium is the same irrespective of whether it is the Marshallian or Hicksian demand function we're using. Thus from (21) we get:

(23) *Roy's identity*: $$H_i(p,u) = D_i(p,M) = -\frac{\partial u^*/\partial p_i}{\partial u^*/\partial M},$$

The importance of Roy's identity is that it allows us to work back from the indirect utility function to the Marshallian demands. This completes the links between the various functions in Fig 1.

2.5. The Slutsky equation

The Slutsky equation, which tells us *inter alia* that the effect of a price change can be decomposed into a substitution effect and an income effect.

The equation is easily derived using the results we have obtained so far. Since the budget constraint binds, M is equal to total expenditure. Hence we can write (22)

(24) $$x_i^* = H_i(p,u) = D_i(p,m(p,u)),$$

Differentiating this with respect to p_j, allowing expenditure to change so as to keep utility constant, we get

(25) $$\frac{\partial H_i}{\partial p_j}=\frac{\partial D_i}{\partial p_j}+\frac{\partial D_i}{\partial M}\frac{\partial m}{\partial p_j} \quad i,j=1,\ldots,n$$

The final term is equal to $H_j(p,u)=x_j$ by Shepherd's lemma. Substituting this in (25) and rearranging gives us:

(26) $$\frac{\partial D_i}{\partial p_j}=\frac{\partial H_i}{\partial p_j}-x_j\frac{\partial D_i}{\partial M} \quad i,j=1,\ldots,n,$$

which is *Slutsky's equation*.

Consider first the case where $i=j$ — i.e. we are considering the effect of a change in p_i on the demand for x_i. In this case, eqn (26) simply says that the slope of the Marshallian demand function is the sum of the *substitution effect*, $\partial H_i/\partial p_i$, and the *income effect*, $-x_i\,\partial D_i/\partial M$. What can we say about the signs of these terms? The first we can sign unambiguously. Recall Shephard's lemma above. Differentiating this with respect to p_i gives:

$$(27) \qquad \frac{\partial^2 m(p,u)}{\partial p_i^2} = \frac{\partial H_i}{\partial p_i},$$

which, given that the expenditure function is concave, means that $\partial H_i / \partial p_i \leq 0$. Thus the substitution effect cannot be positive. The sign of the second term depends on whether good i is normal ($\partial D_i / \partial M > 0$), in which case the Marshallian demand curve is definitely downward-sloping, or inferior ($\partial D_i / \partial M < 0$), in which case it could be downward-sloping, upward-sloping or flat, depending on the relative sizes of the two terms on the RHS of (26). The Slutsky equation can be expressed in elasticity form. Keep $i=j$ and multiply (26) through by p_i/x_i and the second term on the RHS by M/M to get:

$$(28) \qquad \varepsilon_{ii} = \sigma_{ii} - s_i \eta_i,$$

where ε_{ii} is the Marshallian demand elasticity, σ_{ii} is the Hicksian or compensated elasticity, $s_i = p_i x_i / M$ is the share of good i in total expenditure (i.e. its *budget share*) and η_i is the income elasticity of demand. Thus the gap between the Marshallian and Hicksian demand elasticities will be smaller the smaller is the income elasticity of demand and the smaller is the good's budget share.