1 Consumer Theory

- **Theoretical Context**
- Methodological Individualism
- Utilitarianism

Two approaches:

1. could specify a utility function and examine its properties
2. could provide a set of axioms of choice based on preference and indifference

Consider a choice set and a consumption bundle

$$x = (x_1, x_2, ..., x_n)$$

- **Axiom 1:** Reflexivity: each bundle is as good as itself: $x \gtrsim x$ for any bundle
- **Axiom 2:** Completeness: Any two bundles can be compared and either preferred or indifferent to: $x_1 \gtrsim x_2$ or $x_2 \gtrsim x_1$
- **Axiom 3:** Transitivity or consistency if $x_1 \gtrsim x_2$ and $x_2 \gtrsim x_3$ then $x_1 \gtrsim x_3$.

  These axioms define a preference ordering, but not all of these can be represented as utility function and need an indifference surface. This is not possible if discontinuities exist in preferences. The most famous is lexicographic preference ordering eg if $x_1$contains more food than x_2 then it may be preferred to x_3 regardless of what else the bundles contain. Food comes first and clothing second, there are no points of indifference and the utility function can't exist.While reasonable tend to rule out by

- **Axiom 4:** Continuity:For any bundle $x_1$define $A(x_1)$to be the "at least as good" set and $B(x_1)$the "better than" x_1 set

  $$A(x_1) = \{x \mid x \gtrsim x_1\}$$

  $$B(x_1) = \{x \mid x_1 \gtrsim x\}$$

  then A and B are close sets that contain their own boundaries for any x_1 in the choice set

  These 4 axioms allow us to represent the preference ordering by a utlity function $U(x)$ so $U(x_1) \gtrsim U(x_2)$ is equivalent to $x_1 \gtrsim x_2$

  Now we can use $U(x)$ to provide the solution to the solution to the optimisation problem, maximising utility subject to a budget constraint, but we need to have the choices within the budegt set and so:

- **Axiom 5:** Nonsatiation: Utility function $U(x)$ is non decreasing in each of its arguments and $\forall x$ in the choice set is increasing in at least one

  These axioms mean there is a utility fuction that can represent the preference orderings, but it is not necessarily unique. The utility function is defined up to a monotonic increasing transformation, it is ordinal as opposed to cardinal and only orders bundles. The differnces between them are not relevant -cant make interpersonal comparisons.

  Need another axiom to make sure well behaved for optimisation

- **Axiom 6:** Convexity: $x_1 \gtrsim x_0$ then for $0 \leq \lambda \geq 1$

  $$\lambda x_1 + (1 - \lambda)x_0 \gtrsim x_0$$

  from Axiom 4 the "as good as" set $A(x)$ is a convex set. Its indifference curves are convex to the origin.(Distinguish strict convexity $0 < \lambda > 1$which is more stringent as if the consumer has a tiny amount of 2 they will be preared to trade an infinite amount of 1 to gain one uit of the scarce good.) But needed to have unique solution.

- **Axiom 7:** Differentiability: that utility function is differentiable to any required order

  Consider a utility function $U(x) = u^0$ which is strictly quasi concave. we also need to assume differentiability (uniquely defined slope at each point) so we can define a derivative:

  $$\frac{dx_2}{dx_1} |_u = \lim_{\Delta x_1 \to 0} \left(\frac{\Delta x_2}{\Delta x_1} \right)$$

  we define the marginal rate of substitution as:

  $$MRS_{21} = -\frac{dx_2}{dx_1} |_u$$

  the negative sign is there as want a positive value and the slope of the indifference curve is negative

  Now differentiability implies:

  $$du = u_1 dx_1 + u_2 dx_2 + ... + u_n dx_n = 0$$

  where u_i is the partial derivative wich is also the marginal utility

  $$u_i = \frac{\partial u}{\partial x_i}$$

  This means

  $$MRS_{21} = -\frac{dx_2}{dx_1} |_u = \frac{u_1}{u_2}$$

  and the ratios of the marginal utilities are invariant to permissable transformations of the utility function

- Need to consider the feasible set: what is affordable based on the budget constraint. Assume given money income, constant prices and that cannot consume negative amounts

$$p_1x_1 + p_2x_2 + ... + p_nx_n = \sum p_ix_i \leqslant M$$

$$x_i \geqslant 0 \forall i$$

the feasible set is bounded from below by non negativity assumption, closed, convex and non-empty

- Consumption decision combines utility maximisation and the budget constraint -constrained optimisation.

  In the case of 2 goods it is easy to illustrate as the tangent between the highest indifference curve and the budget line. That is

$$\frac{dx_2}{dx_1} |_u = \frac{dx_2}{dx_1} |_M$$

$$MRS_{21} = \frac{u_1}{u_2} = \frac{p_1}{p_2}$$

  or the marginal utility of expenditure on $x_1$is that of x_2 and is the marginal utility of money

$$\frac{u_1}{p_1} = \frac{u_2}{p_2} = u_m$$

  There are possible corner solutions

- More formally assuming a non null solution exists $x^* \gtrsim 0$ to prevent corner solutions

$$\max u(x_1, ..., x_n) \qquad \text{s.t}\ . \sum_k p_kx_k = M, \quad x_i \geqslant 0$$

  so set up Lagrangean

$$L = u(x_1, ..., x_n) + \lambda \left[M - \sum_k p_kx_k \right]$$

$$\begin{aligned} \frac{\partial L}{\partial x_k} &= u_k - \lambda p_k = 0 \\ \frac{\partial L}{\partial \lambda} &= M - \sum_k p_kx_k^* = 0 \end{aligned}$$

  can show

$$\frac{u_i}{p_i} = \lambda$$

which means that the Lagrange multiplier is the marginal utility of money -the shadow price. λ is the rate at which utility increase as money income increases

$$\frac{du^*}{dM} = u_M$$

- Simple comparative statics:

  Can write solution to optimisation problem as determining the demand curve

  $$x^* = D_i(p_1, ..., p_n, M) = D_i(p, M) \quad i = 1, ...n$$

  These are the Marshallian demand curves and and from this can consider the effects of changes in exogenous variables such as income and price. We can derive:

  - income consumption curve (Engle)
  - Price consumption curve

- Results of such analysis is to show the effects of changes in price can result in any outcome. But their are a number of things going on and we can undertake a conceptual exercise to decompose te overall effect of a price change into an income effect and own substitution effect

  - Use compensating variation -change in money income M so as to make the consumer just as well off after a price fall as before (constant real income): Hicksian
  - Alternatively constant purchasing power, unchanged real income: Slutsky -requires less knowledge of indifference map

- Predictions:

  - Normal good cannot be Giffen good: inferiority necessary but not sufficient contdition for Giffen good
  - Own substitution effect always opposite sign to price effect

  But doesn't yield refutable predictions about effect of change in p and M

  - Predicts consumer will not suffer from money illusion D is homogenous of degree zero

    $$x_i^* = D_i(kp, kM) = k^0 D_i(p, M) = D_i(p, M) \quad i = 1, ...n$$

- To make predictions that are testable need to make more specific assumptions about the preferences or the nature of the utility function

  - Common example is additive separability:

$$u\left(x_1,...,x_n\right)=u(x_1)+...+u(x_n)$$
$$\frac{u_1'}{p_n}=\frac{u_2'}{p_n}=...=\frac{u_n'}{p_n}=u_M'$$

where u_1' is the first derivative of $u_i(x_i)$ the marginal utility of good i

This means that the demand for all goods will rise if income rises, there will be no inferior goods

- Have considered the consumer problem as maximising utility subject to budget, but we can also look at this as minimising expenditure necessaty to achieve a specific utility

$$\begin{aligned}\min\sum p_i x_i \quad \text{s.t.... } u\left(x_1,...,x_n\right) &\geqslant u^0\\ x_i &\geqslant 0\forall i=1,...n\end{aligned}$$

if we have strictly positive x_is we can use the Lagrangian

$$L=\sum p_i x_i+\lambda(u^0-u\left(x_1,...,x_n\right))=0$$

first order conditions are

$$\begin{aligned}\frac{\partial L}{\partial x_i} &= p_i-\lambda u_i=0\\ \frac{\partial L}{\partial \lambda} &= u^0-u\left(x_1,...,x_n\right)=0\end{aligned}$$

So

$$\begin{aligned}p_i-\lambda u_i &= 0\Longrightarrow p_i=\lambda u_i\Longrightarrow\lambda=\frac{p_i}{u_i}\\ \text{and}\quad p_j &= \lambda u_j\Longrightarrow\lambda=\frac{p_j}{u_j}\\ \text{so}\ \frac{p_i}{P_j} &= \frac{u_i}{u_j}\end{aligned}$$

We is the same result we got previously. Essentially the expenditure minimising problem is moving along the the indifference curve until the lowest expenditure line is reached

- Now the optimal x is

$$\begin{aligned}x_i^* &= H_i(p_1,...,p_n,u^0)\\ &= h_i(p,u^0)\end{aligned}$$

giving a constant utility Hicksian (compensated) demand function, the slope of which is the substitution effect of a price change

$$\sum p_i x_i^*=\sum p_i H_i(p,u^0)=M(p,u^0)$$

which is the expenditure function, showing the minimum level of expenditure necessary to achieve a given level of utility as a function of prices and required utility level.

- Considering the comparative statics, the effect on m and x^* of changes in price:

$$\begin{aligned} \frac{\partial m}{\partial p_i} &= \frac{\partial}{\partial p_i}\left(\sum_j p_j x_j^*\right) \\ &= \sum_j p_j \frac{\partial x_j^*}{\partial p_i} + x_i^* \end{aligned}$$

Now we know $p_i = \lambda u_i$ meaning

$$\sum_j p_j \frac{\partial x_j^*}{\partial p_i} = \lambda \sum_j u_j \frac{\partial x_j^*}{\partial p_i}$$

so when p_i varies the x_j^*must vary such that the utility level is maintained at u^0

$$\frac{du}{dp_i} = \sum_j u_j \frac{\partial x_j^*}{\partial p_i}$$

so now

$$\frac{\partial m}{\partial p_i} = x^* = H_i(p, u^0)$$

meaning that the partial derivative of the expeniture function wrt price of a good is the constat utility demand for that good.

Now $\frac{\partial H_i}{\partial p_i}$is the own substitution effect and $\frac{\partial H_i}{\partial p_j}$ the cross substitution effect and we can show that these are equal.

We can also show that the own substitution effect is negative (the expenditure function is concave)

$$\frac{\partial^2 m}{\partial p_i^2} = \frac{\partial h_i}{\partial p_i} < 0$$

- An alternative way of analysing the consumer problem is through the **indirect utility function**

From before we are solving

$$\max u\left(x_1, ..., x_n\right) \quad \text{s.t} . \sum_k p_k x_k \leqslant M, \text{ and } \ x_i \geqslant 0$$

and the optimal x_i are

$$x_i^* = D_i(p_1, ..., p_n, M) = D_i(p, M) \quad i = 1, ...n$$

So the maximised values of $u(x_1, ..., x_n)$ will also be functions of $p_i$and M

$$\begin{aligned} u(x_1^*, ..., x_n^*) &= u(D_1(p, M), ..., D_1(p, M)) \\ &= u^*(p, M) \\ &= u^*(p_1, ..., p_n, M) \end{aligned}$$

u^* is the indirect utility function, in the sense that utility depends indirectly on prices and money income via the maximisation process.

- Consider the effects of changes in price and income on utility:

$$\frac{\partial u^*}{\partial M} = u_1 \frac{\partial x_1^*}{\partial M} + ... + u_n \frac{\partial x_n^*}{\partial M} = \sum u_i \frac{\partial x_i^*}{\partial M}$$

from the previous Lagrangian we know $u_i = \lambda p_i$ so

$$\frac{\partial u^*}{\partial M} = \lambda \sum p_i \frac{\partial x_i^*}{\partial M}$$

when M changes the budget constraint is still staisfied so

$$\sum p_i x_i = M$$

hence

$$\begin{aligned} \frac{d \sum p_i x_i}{dm} &= \frac{dM}{dM} \\ \text{or } \sum p_i \frac{\partial x_i^*}{\partial M} &= 1 \end{aligned}$$

so

$$\frac{\partial u^*}{\partial M} = \lambda$$

which is the result discussed previously with λ the marginal utility of money

- Considering the effect of a change in price:

$$\frac{\partial u^*}{\partial p_i} = \sum u_k \frac{\partial x_k^*}{\partial p_i} = \lambda \sum p_k \frac{\partial x_k^*}{\partial p_i}$$

budget constraint requires

$$\sum p_k x_k = M$$

so

$$\frac{d\sum p_k x_k^*}{dp_i} = \frac{dM}{dp_i} = 0$$

so

$$\begin{aligned}
\sum p_k \frac{\partial x_k^*}{\partial p_i} + x_i^* &= 0 \\
\text{or } \sum p_k \frac{\partial x_k^*}{\partial p_i} &= -x_i^* \\
\text{so } \frac{\partial u^*}{\partial p_i} &= -\lambda x_i^* = \frac{\partial u^*}{\partial M} x_i^*
\end{aligned}$$

which is known as Roy's identity

Intuitively the second term is the rate at which utility varies with money income times the rate at which at which (the purchasing power of) money income varies with p_i. ie the change of u wrt p

As $\lambda > 0$ Roy's identity means an increase in the price of a good reduces maximised utility the larger the quantity bought

- A major advantage of this approach is that it makes it much easier to get results from the consumer problem and helps in the analysis of demand functions. In particular, the Slutsky equation plays a central role.

  Consider

$$x_i^* = H_i(p, u^0) = D_i(p, M)$$

  substiute for u^0

$$x_i^* = H_i(p, u^*(p, M)) = D_i(p, M)$$

  Before with expenditure minimising we hold u^0 constant as p_j varies

  Here we allow u^0 to vary with u^* because we want to look at the effects of changes in p_j on utility maximising demand for x_i

$$\frac{\partial h_i(p, u^*(p, M))}{\partial p_j} = \frac{\partial H_i}{\partial p_j} + \frac{\partial H_i}{\partial u^0} \cdot \frac{du^0}{du^*} \cdot \frac{\partial u^*}{\partial p_j} = \frac{\partial D_i}{\partial p_j}$$

  - $\frac{\partial H_i}{\partial p_j}$is the direct effect of a change in the relative prices with utility constant
  - $\frac{\partial H_i}{\partial u^0}$is the rate at which demand for $x_i$varies when the required utility level changes with prices constant
  - $\frac{du^0}{du^*}$is the rate at which the required utility levels changes as the max level of utility changes

- $\frac{\partial u^*}{\partial p_j}$ is the rate at which maximised values of utility change as p_j increase

  Can simplify by differentiating with respect to M

$$\frac{\partial H_i(p, u^*(p, M))}{\partial M} = \frac{\partial H_i}{\partial u^0} \cdot \frac{du^0}{du^*} \cdot \frac{du^*}{dM} = \frac{\partial D_i}{\partial M}$$

  and remebering from before that

$$\frac{du^*}{dp_j} = -\lambda x_j \text{ and } \frac{du^*}{dM} = \lambda$$

which means

$$\begin{aligned} \frac{\partial H_i}{\partial u^0} \cdot \frac{du^0}{du^*} \cdot \frac{du^*}{dp_j} &= -\frac{\partial H_i}{\partial u^0} \cdot \frac{\partial D_i}{\partial M} \cdot \lambda x_j \\ &= -\frac{\partial H_i}{\partial u^0} \cdot \frac{du^0}{du^*} \cdot \frac{du^*}{dM} \cdot x_j \\ &= -\frac{\partial D_i}{\partial M} \cdot x_j \end{aligned}$$

so

$$\frac{\partial H_i}{\partial p_j} - \frac{\partial D_i}{\partial M} \cdot x_j = \frac{\partial D_i}{\partial p_j}$$

- $\frac{\partial D_i}{\partial p_j}$ is the price effect which is broken down into
- $\frac{\partial H_i}{\partial p_j}$ the substitution effect and
- $\frac{\partial D_i}{\partial M} \cdot x_j$ the income effect, composed of the effect of the change in M on D_i and the rate at which the purchasing power of money income changes as price changes

Cost function	←	*inversion*	→	Indirect Utiltiy function
↓	↘		↙	↓
Differentiate		↙ ↘		*Roys Identity*
↓	↙		↘	↓
Hicksian Demand	←	*substitution*	←	Marshallian Demand
	⟶		→	

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