0.1 Production

- Exchange economy: endowments and trade
  - Problem of production:
  - Problem of existence of firms -firms missing
  - Ignore issues for now
- Production Theory: Starting point is minimisation of cost of a given level of output subject to technological constraints. consider 2 inputs z_i $i = 1, 2$ to produce a single output y so $y = f(z_1, z_2)$
  - Assume

1. (a) i. Inputs and outputs divisible
   ii. Corresponding to each combination of inputs there is a maximum output $y_{\max}$
   iii. Wasteful production is possible $0 \leqslant y \leqslant y_{\max}$.
   iv. $y > 0$ implies either $z_1 > 0$ or $z_2 > 0$
   v. Production function is twice differentiable. A relatively strong assumption.
   vi. The marginal productivity of at least one input is always positive
   vii. For any output level the set of input pairs that produce it is strictly convex ie production function is strictly quasi concave

- MP_i = rate at which max output of y changes in response to a change in z_i

$$MP_i = \frac{\partial f(z_1, z_2)}{\partial z_i} = f_i(z_1, z_2) = f_i$$

second order partial is the rate at which the marginal products changes as one input is varied

$$\frac{\partial^2 f}{\partial z_1 z_2} = f_{12}$$

we have assumed these exist and are continuous functions and that $f_{12} = f_{21}$ie continuous and defined at every point

- Analogously to consumer theory we can define an isoquant as the inputs that produce a specified output level when used efficiently

$$I_0 = \left\{(z_1, z_2) \mid f(z_1, z_2) = y_{\max} = y^0\right\}$$

Now take

$$y_{\max} = f(z_1, z_2) = y^0$$

differentiate totally

$$dy_{\max} = f_1 dz_1 + f_2 dz_2 = 0$$

which constrains changes in $z_1$and z_2 to be the same as move along the isoquant

$$-\frac{dz_1}{dz_2}\mid_{y_{\max}=y^0}=\frac{f_1}{f_2}=\frac{MP_1}{MP_2}$$

this is $MRTS_{12}$ and is analogous to MRS though cardinal rather than ordinal, but indep of units in whch measure output

- Cost minimisation -choose input combinations to minimise the cost of producing specified output level. The feasible set Z contains the combinations thet produce at least the desired y

  $$Z=\left\{(z_1,z_2)\mid f(z_1,z_2)\geqslant y^0\right\}$$

  with efficient production when $f(z_1,z_2)=y^0$ and inefficient when $f(z_1,z_2)>y^0$

  - Consider isoquants and ridge lines...
    - ∗ On or inside them is the economic region and the cost min firm will always choose point within
    - ∗ For any on I^0 point outside always point inside with less of both inputs
    - ∗ Though input comb ouptput efficient it may not be technically efficient -economic region is technically efficient
  - Firm can produce changes in output by:
    - ∗ Varying all inputs and so changing the scale of production
    - ∗ Varying the relative input proportions

- In isoqant diagram changes in scale moves along a ray through origin while changes in relative input proportions moves to new isoquant on a new ray

  - Unit isoquant is the set of all input combinations that produce one unit of output: I_0.

    $$z^0=(z_1^0,z_2^0)\text{ is on }I_0\text{ and }f(z_1^0,z_2^0)=1$$

  - Consider scale parameter s with

    $$y=f(sz_1^0,sz_2^0)=y(s)$$

    if $s<1$ move along ray towards origin and if $s>1$ move away from the origin

- Elasticity of production

  $$E=\frac{dy}{y}\frac{s}{ds}=\frac{dy}{ds}\frac{s}{y}$$

  the responsiveness of output to equal proportionate change in all inputs

- Increasing return to scale: $E > 1$
- Constant returns to scale: $E = 1$
- Decreasing returns to scale: $E < 1$

- Homogeneous production functions

$$f(sz_1, sz_2) = s^n \cdot f(z_1, z_2) = s^n \cdot y$$

means homogeneous of order n

  - When $n = 1$linear homogeneous which is a popular assumption as it has a number of useful properties
    - It means

$$y = f(z_1, z_2) = f(sz_1^0, sz_2^0) = sf(z_1^0, z_2^0) = s \text{ as } f(z_1^0, z_2^0) = 1$$

which gives

$$E = \frac{dy}{y}\frac{s}{ds} = 1$$

implying contant returns at all points on rays

  - The MP of an input depends only on relative input proportions and not absolute levels

$$\begin{aligned} f(sz^0) &= sf(z^0) \Longrightarrow \frac{\partial f(sz^0)}{\partial z_i} = \frac{\partial sf(z^0)}{\partial z_i} \\ \text{as } z &= sz^0 \Longrightarrow \frac{\partial f(z)}{\partial z_i} = \frac{\partial f(z^0)}{\partial z_i} \end{aligned}$$

A function that is homogeneous of degree n has partial derivatives that are $n - 1$ so the partial derivatives of linear homogeneous are 0 meaning the MPs are independent of scale (f_1 and f_2 are constant along ray)

  - Mean slopes of isoquants $-\frac{f_1}{f_2}$ depend only on the input ratio, so isoquants are radial expansions. Will see this means input props of cost min firm depend only on input prices and not level of output
  - Take $f(sz^0) = sf(z^0)$ remembering $f(z^0) = 1$then

$$\begin{aligned} \frac{dy}{ds} &= \sum_i f_i(z)\frac{dz_i}{ds} = \sum_i f_i(sz^0)\frac{dsz_i^0}{ds} = \sum_i f_i(sz^0)\frac{dsz_i^0}{ds} \\ \text{Now}\frac{ds^n f(z)}{ds} &= ns^{n-1}f(z) \Longrightarrow \text{if } n = 1 \text{ then}\frac{dsf(z)}{ds} = f(z) \text{ so}\frac{dsz_i^0}{ds} = z_i^0 \\ &\Longrightarrow \frac{dy}{ds} = \sum_i f_i(sz^0)z_i^0 = s^{n-1}\sum_i f_i(z^0)z_i^0 \\ &\Longrightarrow \sum_i f_i(z^0)z_i^0 = nf(z^0) = f(z^0) \end{aligned}$$

which give the adding up property that the output is equal to the sum of the marginal products of input times the level of their use. Means if price of each input is value of MP then revenue will equal costs.

- Can generalise to multi product case -see text
- Reminder: Production function is homothetic if $f(z)$can be written as $h(g(z))$where h is monotonic and g is homogeneous of degree 1

  $f(z)$ is a monotonic transformation of a function that is homeogenous of degree zero, it is a non-homogeneous function that shares property that $\frac{\partial f(z)}{\partial z_i}$ is homogeneous of degree zero
- Reminder: Eulers Law: if f is a differential homog of degree 1 function, then

  $$f(z) \equiv \sum_{1=1}^{n} \frac{\partial f(z)}{\partial z_i} z_i$$

  For proof note $f(tz) \equiv tf(z)$ and differentiate identity wrt t

  $$\sum_{1=1}^{n} \frac{\partial f(tz)}{\partial z_i} z_i = f(z)$$

  and set$t = 1$ to get the result

0.2 Costs

- So far have ignored time dimension but output is a flow and while inputs can be flows they will also be durable inputs. This means stocks of productive services and capacity of assets have to be allowed for
- Need to consider
  - long and short run decision makein
  - adjustment costs
  - opporunity costs
- Long run cost minimisation

  $$\begin{aligned} & \min \sum p_i z_i \\ st\ y &= f(z_1, ..., z_n) \geqslant y^0 \\ z_i &\geqslant 0 \end{aligned}$$

for 2 inputs can draw isocost lines to superimpose on isquants and find tangent

$$\begin{aligned} p_1 z_1 + p_2 z_2 &= c_1 \\ z_2 &= \frac{c_1}{p_2} - \frac{p_1}{p_2} \cdot z_1 \\ \frac{dz_2}{dz_1} & \mid_{dc=0} = -\frac{p_1}{p_2} \end{aligned}$$

where prices indep of amounts , isocost line parallel.

Slope of the isoquant is negative of $MRTS$

$$-\frac{p_1}{p_2} = \frac{dz_2}{dz_1} \mid_{y=y^0} = -MRTS_{21} = -\frac{f_1}{f_2}$$

so ratio of input prices is equal to the ratio of the MPs

$$\frac{p_1}{f_1} = \frac{p_2}{f_2} = LMC$$

necessary condition for cost minimisation -long run marginal cost (LMC)

- Economic efficiency: input combination that minimises cost of producing a given output.
- Generalising the above

$$\begin{aligned} & \min \sum p_i z_i \\ st\ y &= f(z_1, ..., z_n) \geqslant y^0 \\ z_i &\geqslant 0 \end{aligned}$$

$$\begin{aligned} L &= \sum p_i z_i + \lambda[y^0 - f(z_1, ..., z_n)] \\ \frac{\partial L}{\partial z_i} &= p_i - \lambda f_i = 0 \\ \frac{\partial L}{\partial \lambda} &= y^0 - f(z_1, ..., z_n) = 0 \\ &\Longrightarrow \frac{p_i}{p_j} = \frac{f_i}{f_j} \end{aligned}$$

λ is that rate at which optimised value of the objective function increases as the constraint parameter is increased, the rate at which cost increases as output increases =LMC

$$\lambda = \frac{\partial c}{\partial y^0} = LMC$$

where c is the minimum value of $\sum p_i z_i$

- Effects of output change:

  consider optimal input conditions

$$z_i^* = z_i^*(p_1, ..., p_n, y) = z_i^*(p, y)$$

  minimise total cost:

$$C = \sum_i p_i z_i^*(p, y) = C(p, y)$$

  both function of input prices and output

- Comparative statics: no predictions can be made without knowledge of the form of the production function.

  If linear homogeneous -necessary condition for cost minimisation - then expnasion path is a straight line and only changes in prices will lead to changes in the ratio of inputs.

- Long run cost curves: relation between total cost and output

$$\begin{aligned} \frac{\partial LTC}{\partial y}.\frac{y}{LTC} &= \frac{LMC}{LAC} \\ \text{diseconomies} &: \frac{LMC}{LAC} > 1 \\ \text{Economies} &: \frac{LMC}{LAC} < 1 \end{aligned}$$

  between these two ranges the cost function shape changes, with $\frac{LMC}{LAC} = 1$ at the lowest point on the LAC curve. The actual point depends on the shape of the production function as assume input prices constant

- Effects of input price changes

  - input substitution effect as in consumer theory but can't say mich without knowing the shape of the production function
  - Can define elasticity of substitution: cost minisation give $MRTS$= input price ratio

$$\sigma = \frac{d(z_2/z_1)}{d(p_2/p_1)} \cdot \frac{p_1/p_2}{z_1/z_2}$$

  - input price changes and cost curves
  - Economies of scale and indivisible outputs -types of models

- Short run cost minimisation

  - Choose (z_1, z_2) to min cost of given output when constraints on adjustment of z_2;
    - firm faces quota on z_2, then only pays for what it uses

firm constrained to pay $p_2 z_2^0$ for fixed input regardless o f amount it actually uses ie there is a fixed cost

- Short run average costs and marginal costs
  - Derived similarly to long run

$$\begin{aligned} S(y) &\geqslant C(y) \\ &\Longrightarrow \frac{S(y)}{y} \geqslant \frac{C(y)}{y} \end{aligned}$$

    so SRAC never less than $LRAC$ and $SRAC$ is tangent to $LRAC$ because $S(y)$ is tangent to $C(y)$

  - Differentiate $SRAC$ wrt y

$$\frac{dSRAC}{dy} = \frac{1}{y^2}\left[\frac{dS}{dy}y - S\right]$$

    but at y^0

$$\frac{dS}{dy} = \frac{dC}{dy} \text{ and } S = C$$

    in the case of two inputs

$$\begin{aligned} S &= VC + FC = p_1 z_1 + p_2 z_2 \\ AVC &= \frac{p_1 z_1}{y} = \frac{p_1}{AP_1} \\ SMC &= \frac{p_1}{f_1} = \frac{p_1}{MP_1} \end{aligned}$$

    by similar arguments to the long run case

  - Diagrammatically...

- Envelope property: $LRAC$ can be seen as "envelope" of the $SRAC$ curves, but $LRMC$ not envelope of MC.
  - costs more in SR to expand output than in LR when can improve by changing capital
  - SR input combinations converge to LR ones

0.3 Using Production Functions

- Productions functions are an important component of applied economic analysis:
  - * **Macro level:** combined with MP theory to explain prices of factors of production and the extent to which utilised. Important for growth/distribution

    * **Micro level:** used to investigate substitutability between factors returns to scale
    * **Both:** used to consider what proportion of growth is the result of increase in factor inputs, returns to scale and technical progress

- Have been object of considerable controversy -Cambridge capital controversy - but remain an important component of applied economics and with the resurgence of growth theory in the form of new growth theory are increasingly important.
- Basic model

$$Q = Q(K, L)$$

  Defines the maximum output Q given inputs of capital K and labour L.
- Note these are flow variables and we assume variable/divisible and continuously substitutable.
- Technical questions of how to get from K and to the best Q is assumed solved, but substitution means can get a given from a number of combinations and so need to consider the minimum cost combination

$$\begin{aligned} \frac{\partial Q}{\partial L} &= MP_L \\ \frac{\partial Q}{\partial K} &= MP_K \end{aligned}$$

  which are positive but diminishing marginal productivity: diminishing returns to a factor

$$Q\left(K, L\right) = (\lambda K, \lambda L) = \lambda^n Q(K, L)$$

  homogeneous of degree n, if $n < 1$ decreasing returns,$n = 1$ constant returns $n > 1$ increasing returns.
- Questions of what determines the proportions by which the factors are combined is an economic one.
- At micro level use model of firm behaviour maximising profits π

$$\begin{aligned} \pi &= pQ - mK - wL \\ \text{st } Q &= Q(K, L) \end{aligned}$$

  Assuming perfect competition in production and factor markets the factor prices p, m, w are given and the Lagrangian

$$L = pQ - mK - wL - \lambda[Q - Q(K, L)]$$

- An alternative economic model is to assume Q is predetermined and minimise costs subject to the level of output.

$$\begin{aligned} \min C &= mK + wL \\ \text{st } Q^0 &= Q(K, L) \\ L &= mK + wL - \lambda[Q^0 - Q(K, L) \end{aligned}$$

- Both the profit maximizing and cost minimising models imply that factors are combined so as to equate the MRS with the ratio of factor prices.

$$MRS = \frac{\partial Q}{\partial L} / \frac{\partial Q}{\partial K} = \frac{w}{m}$$

  Even if the assumption of perfect competition is dropped profit maximising still implies this.
- MRS measures the extent it is possible to substitute one factor for another in the production of a given output, but its size will depend on the units of measurement of L and K. For this reason the elasticity of substitution is used

$$\sigma = \frac{d\,(K/L)}{K/L} / \frac{dMRS}{MRS}$$

  The higher σ the more the substitution possibilities.

- Cobb-Douglas Production Function:
  - In applied work the most commonly used form of the production function is the Cobb Douglas. This resulted from Douglas observing that the share of national output going to labour was approximately constant over time

$$wL = \beta pQ$$

  - An underlying production function that would give rise to this observation is of the form

$$Q = AK^{\alpha} L^{\beta}$$

  - This has a number of convenient properties α, β are the elasticities of output and A can be considered an efficiency parameter

$$\begin{aligned} \frac{\partial Q}{\partial K} &= \alpha A K^{\alpha - 1} L^{\beta} = \alpha \frac{Q}{K} \\ \frac{\partial Q}{\partial L} &= \alpha A K^{\alpha} L^{\beta - 1} = \beta \frac{Q}{L} \end{aligned}$$

  Assuming the firm is a price taker and profit maximiser then

$$\begin{aligned} \frac{\partial Q}{\partial K} &= \alpha \frac{Q}{K} = \frac{m}{p} \\ \frac{\partial Q}{\partial L} &= \beta \frac{Q}{L} = \frac{w}{p} \end{aligned}$$

which can be written as:

$$\alpha = \frac{mK}{pQ}$$
$$\beta = \frac{wL}{pQ}$$

which is the regularity Douglas observed.

- So if the MP conditions hold then α and β in the C-D are equal to the respective shares of capital and labour in the share of national output (also need constant returns to scale). As before this result can be derived from a cost minimising approach and for both models the optimising conditions imply:

$$\frac{K}{L} = \left(\frac{\alpha}{\beta}\right)\left(\frac{w}{m}\right)$$

For any given factor price ratio the greater is α/β the greater is K/L. So size of α relative to β determines capital intensity of the production process represented by the C-D

- C-D is also homogeneous of degree $\alpha + \beta$

$$Q(\lambda K, \lambda L) = A(\lambda K)^{\alpha}(\lambda L)^{\beta} = \lambda^{\alpha+\beta} A K^{\alpha} L = \lambda^{\alpha+\beta} q(K, L)$$

  * $\alpha + \beta > 1$ increasing returns to scale
  * $\alpha + \beta = 1$ constant returns to scale
  * $\alpha + \beta < 1$ decreasing returns to scale

- NB the returns to scale property is the same at all levels of output and the C-D implies a constant elasticity of substitution that is equal to one

$$\sigma = \frac{d(K/L)}{K/L} / \frac{d(w/m)}{w/m}$$

- - There is a problem that for the first order solutions to be unique decreasing return to scale are required, but is unlikely and if it doesn't hold can get some strange results. The answer is to relax the assumptions of perfect competition, where prices are exogenously given, but this make prices endogenous and suggests product demand and factor supply equations should be added to the system.
  - Note that we can also use the framework to look at factor demands and productivity. Consider:

$$Y = AK^{\alpha}L^{\beta}$$

to operationalise this we normally take logs and estimate

$$\log Y = \log A + \alpha \log K + \beta \log L$$

using lower case for logs we can rewrite as

$$y = \alpha_0 + \alpha_1 k + \beta l$$

assuming constant returns to scale $\alpha_1 + \beta = 1$ so

$$\begin{aligned} y &= \alpha_0 + \alpha_1 k + (1 - \alpha_1) l \\ y &= \alpha_0 + \alpha_1 k + l - \alpha_1 l \end{aligned}$$

- CES Production Function
  - A more general, though also more complex production function is one with constant but not necessarily unity elasticity of substitution, the CES production function. Arrow, Chenery,Minhas and Solow (1961) estimated cross section equations

    $$\frac{Q}{L} = \frac{1}{\beta}\left(\frac{w}{p}\right)^{\chi}$$

  - now a C-D with profit maximising and perfect competition would imply $\chi = 1$, but they consistently found it to be less, implying that the underlying function must be:

    $$Q = \gamma \left[\delta K^{-\theta} + (1-\delta) L^{-\theta}\right]^{-1/\theta}$$

    where γ is efficiency parameter similar to A in the C-D. This means

    $$\frac{\delta Q}{\delta K} = \frac{\delta \gamma}{K^{1+\theta}} \left[\delta K^{-\theta} + (1-\delta) L^{-\theta}\right]^{-\frac{(1+\theta)}{\theta}} = \frac{\delta}{\gamma^{\theta}} \left(\frac{\theta}{K}\right)^{1+\theta}$$

    $$\frac{\delta Q}{\delta K} = \frac{(1-\delta)\gamma}{K^{1+\theta}} \left[\delta K^{-\theta} + (1-\delta) L^{-\theta}\right]^{-\frac{(1+\theta)}{\theta}} = \frac{(1-\delta)}{\gamma^{\theta}} \left(\frac{\theta}{K}\right)^{1+\theta}$$

  - Marginal productivity equations under profit maximization and perfect competition are:

    $$\frac{\delta}{\gamma^{\theta}} \left(\frac{\theta}{K}\right)^{1+\theta} = \frac{m}{p}$$

    $$\frac{(1-\delta)}{\gamma^{\theta}} \left(\frac{\theta}{K}\right)^{1+\theta} = \frac{w}{p}$$

    which leads to the SMAC estimating equation:

    $$\frac{Q}{L} = \frac{1}{\beta}\left(\frac{w}{p}\right)^{\chi} \quad \text{with} \quad \chi = \frac{1}{1+\theta} \text{ and } \frac{1}{\beta} = \left(\frac{\gamma^{\theta}}{1-\delta}\right)$$

– MRS is

$$\left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{1+\theta}$$

– Elasticity of substitution is:

$$\sigma = \frac{1}{1+\theta}$$

θ is known as substitution parameter as $\theta = \frac{1}{\sigma} - 1$

* when $\theta = \infty$ then $\sigma = 0$ and substitution is impossible
* when $\theta = -1$ then $\sigma = \infty$ and isoquants are straight lines
* when $\theta = 0$ then $\sigma = 1$ and we have the C-D production function

– Can also write

$$\frac{wL}{mK} = \left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{\theta}$$

So for a given K/L and θ, δ will determine the shares of capital and labour. It is the "distribution parameter". For the C-D the ratio of factor shares was constant.

– Note that the CES above implies constant returns to scale, but we can generalise to

$$Q = \gamma\left[\delta K^{-\theta} + (1-\delta)L^{-\theta}\right]^{-v/\theta}$$

v is then the returns to scale factor:

* $v > 1$ increasing returns to scale
* $v = 1$ constant returns to scale
* $v > 1$ decreasing returns to scale