## Differentials, slopes, and rates of change

Economics is frequently concerned with marginal effects - Marginal cost, marginal utility, marginal revenue, etc.

When relationships between variables are expressed in functional terms, the marginal effect is the rate of change of the function with respect to the argument. So if costs $\mathrm{C}=\mathrm{C}(\mathrm{Q})$ where Q is output, then marginal costs is the rate of change of the function as Q changes.

This is the same as the slope of a function on a graph.
E.g. If we have a function
$\mathrm{F}(\mathrm{x})=3+2 \mathrm{X}$, with the graph


Each increase of 1 in $X$ leads to an increase in 2 of $F(X)$. The slope of the line is 2 . We could say that the marginal increase in F for a change in X is 2 .

Straight lines are easy, as the slope is always the same - the slope of the above graph is 2 for all values of X .

When we have a non-linear function - for example $F(X)=X^{2}$ - the slope of the graph, and therefore the rate of change of $\mathrm{F}(\mathrm{X})$, varies depending on the value of X .

We can measure the rate of change, or the marginal effect, in two different ways: first, most easily, by looking at the change in X and the change in $F(X)$ between two points:

For example, between $\mathrm{X}=2$ and $\mathrm{X}=3, \mathrm{~F}(\mathrm{X})$ goes from 4 to 9 , so the rate of change is $(9-4) /(3-2)=5$.

But it is more precise to measure the rate of change at a particular point. We do this by looking at the slope of the tangent line to the curve at the point we're interested in.

We can also see that, if we are taking the slope between two points, the nearer these pints are together, the closer the slope is to the slope of the tangent line. .


The rate of change at a particular point X - that is, the slope of the tangent line - is also known as the differential of the function $\mathrm{F}(\mathrm{x})$ at X .

We write the differential at a point X as $\mathrm{F}^{\prime}(\mathrm{X})$. (For example, at $\mathrm{X}=1$, we write the slope F'(1).)

In general, the differential changes as x changes, and so is itself a function of x , written $\mathrm{F}^{\prime}(\mathrm{x})$.

If we have $Y=F(X)$, then we write the differential as $\frac{d Y}{d X}$.
Formally, the differential of $\mathrm{F}(\mathrm{X})$ at the point $\mathrm{X}=\mathrm{a}$ is given by:
$F^{\prime}(\mathrm{a})=\operatorname{Lim}_{X \rightarrow a} \frac{F(X)-F(a)}{X-a}$ Where Lim here means "Limit".

## Rules for differentiation

There are some fairly simple rules for differentiating all the basic functions you are likely to meet in this course.

1) Constant functions: $F(X)=a$, where $a$ is a constant, e.g. $F(X)=3$. These are flat, they have slope 0 , so $F(X)=0$.
2) Linear functions: If $F(X)=a+b X$, then $F^{\prime}(X)=b$. (Linear functions have a constant slope).
3) If $F(X)=X^{n}$, where $n$ is any number (positive or negative, not necessarily an integer (whole number)), then $F^{\prime}(X)=n X^{n-1}$.
4) If $\mathrm{F}(\mathrm{X})=\operatorname{Ln}(\mathrm{X})$ where Ln is the natural logarithm, then $\mathrm{F}^{\prime}(\mathrm{X})=1 / \mathrm{X}$.
5) If $F(X)=e^{X}$ (the exponential function), then $F^{\prime}(X)=e^{X}$.
6) If $F(X)=\sin (X)$, then $F^{\prime}(X)=\cos (X)$. If $F(X)=\cos (X)$, then $F^{\prime}(X)=-\sin (X)$.

For example (case 3), if $Y=X^{2}$, then $d Y / d X=2 X-i n$ other words, the slope increases as X increases, as we can see from the graph.

## Rules for combining functions

1) Addition of functions: If $F(X)=G(X)+H(X)$, then $F^{\prime}(X)=G^{\prime}(X)+$ H'(X)
2) Multiplication by a constant: If a is a constant, then the differential of $\mathrm{aF}(\mathrm{X})$ is $\mathrm{aF}^{\prime}(\mathrm{X})$. (E.g. the differential of $2 \mathrm{X}^{2}$ is $2 * 2 \mathrm{X}=4 \mathrm{X}$ ).
3) Multiplication of functions: If $\mathrm{F}(\mathrm{X})=\mathrm{G}(\mathrm{X}) \mathrm{H}(\mathrm{X})$ then $\mathrm{F}^{\prime}(\mathrm{X})=\mathrm{G}(\mathrm{X}) \mathrm{H}^{\prime}(\mathrm{X})$ + G'(X)H(X).
4) Division of functions: If $F(X)=G(X) / H(X)$, then $F^{\prime}(X)=$

$$
\frac{H(X) G^{\prime}(X)-G(X) H^{\prime}(X)}{(H(X))^{2}}
$$

For example, if $\mathrm{Y}=(\mathrm{X}+3)(3-2 \mathrm{X})$, we let $\mathrm{G}(\mathrm{X})=\mathrm{X}+3$, and $\mathrm{H}(\mathrm{X})=3-2 \mathrm{X}$. Then, $\mathrm{G}^{\prime}(\mathrm{X})=1$, and $\mathrm{H}^{\prime}(\mathrm{X})=-2$. Thus, $\mathrm{dY} / \mathrm{dX}=(\mathrm{X}+3)^{*}(-2)+1^{*}(3-2 \mathrm{X})=-$ $2 \mathrm{X}-6+3-2 \mathrm{X}=-4 \mathrm{X}-3$.
5) Function of a function: If $F(X)=G(H(X))$, then $F^{\prime}(X)=G^{\prime}(H(X)) H^{\prime}(X)$

For example, if $\mathrm{F}(\mathrm{X})=e^{\left(x^{2}\right)}$, we let $\mathrm{G}()=.\mathrm{e}^{(.)}$, and $\mathrm{H}(\mathrm{X})=\mathrm{X}^{2}$.

Now $\mathrm{G}^{\prime}(\mathrm{X})=\mathrm{e}^{\mathrm{X}}$, so $\mathrm{G}^{\prime}()=.\mathrm{e}^{(.)}$, so $\mathrm{G}^{\prime}(\mathrm{H}(\mathrm{X}))=\mathrm{e}^{\mathrm{H}(\mathrm{X})}=e^{\left(\mathrm{x}^{2}\right)}$.

Also $\mathrm{H}^{\prime}(\mathrm{X})=2 \mathrm{X}$, so

$$
\mathrm{F}^{\prime}(\mathrm{X})=\mathrm{G}^{\prime}(\mathrm{H}(\mathrm{X})) \mathrm{H}^{\prime}(\mathrm{X})=e^{\left(x^{2}\right)} * 2 \mathrm{X}
$$

