## Finite and infinite sums

Finite sums: notation

$$
\sum_{i=1}^{n} a_{i} \text { means } \mathrm{a}_{1}+\mathrm{a}_{2}+\ldots+\mathrm{a}_{\mathrm{n}}
$$

Very often, $a_{i}$ will be defined by some formula or function of $i$.
E.g. $\sum_{i=1}^{10} i^{2}$ means $1^{2}+2^{2}+\ldots+10^{2}$. (In this case, $a_{i}=i^{2}$ ).

Here $\mathbf{i}$ is called the index variable, and $\mathrm{a}_{\mathrm{i}}$ the summand, that is the thing being summed.
$\sum_{x \in A} a_{x}$ Where A is a set means we add up $\mathrm{a}_{\mathrm{x}}$ over all values of x belonging to the set A. We may also sum over all values of x satisfying a certain condition. E.g. $\sum_{x=1.10, \text { xodd }} x^{2}$ means $1^{2}+3^{2}+\ldots+9^{2}$. Or, suppose we have a set $A$ of people, and we wish to add up their (say) incomes, we could write $\sum_{x \in A} Y(x)$ Where $Y(x)$ is the income of individual $x$.

Infinite sums: notation
$\sum_{n=1}^{\infty} F(n)$ means the infinite sum $F(1)+F(2)+F(3)+\ldots$.
This may or may not sum to a finite total.
Some tricks for dealing with summations
A particularly common type of sum (finite or infinite) is a geometric progression, where the ratio between each term and the next is a constant. That is, a series of the form
$\sum_{k=0}^{n} a r^{k}$ or $\sum_{k=0}^{\infty} a r^{k}$
(The first a finite, the second an infinite sum). Let's deal with the finite sum first.

Let $S=\sum_{k=0}^{n} a r^{k}=a+a r+a r^{2}+\ldots+a r^{n}$
Then rS =

$$
\mathrm{ar}+\mathrm{ar}^{2}+\ldots+\mathrm{ar}^{\mathrm{n}}+\mathrm{ar}^{\mathrm{n}+1}
$$

So $\mathrm{S}-\mathrm{rS}=\mathrm{a}-\mathrm{ar}^{\mathrm{n}+1}$
So $S(1-r)=a\left(1-r^{n+1}\right)$
Hence $S=\frac{a\left(1-r^{n+1}\right)}{1-r}$ (unless $r=1$, in which case of course the sum is simply $a(n+1)$.

We can see that, if $|\mathrm{r}|<1$, that is if $-1<\mathrm{r}<1$, then the bracketed term at the top will get closer and closer to 1 as $n$ gets larger. On the other hand, if $|r|>1$, then the bracketed term will get larger and larger in magnitude as $n$ increases. This suggests when and how we can calculate the infinite sum.

Let $S=a+a r+a r^{2}+a r^{3}+\ldots$ (infinite sum)
So $\mathrm{rS}=\mathrm{ar}+\mathrm{ar}^{2}+\mathrm{ar}^{3}+\ldots$.
Hence $S(1-r)=a$
Whence $\mathrm{S}=\frac{a}{1-r}$.
However, this sum will only be valid in the case $|\mathrm{r}|<1$.

## Formal definition

Let $\sum_{k=1}^{\infty} a_{k}$ be an infinite sum. We define the n'th partial sum as:
$S_{n}=a_{1}+\ldots+a_{n}-$ the sum of the first $n$ terms of the infinite series.
We say that the infinite sum converges if the sequence of partial sums, $S_{1}$, $S_{2}, S_{3}, \ldots$ tends to some limit $S$. This limit $S$ is then defined to be the sum of the infinite series. Otherwise, we say the infinite sum diverges, and has no value.

In other words, if when we add on successive terms of the series, we get closer and closer to some limiting value (I will not define precisely what is meant by this), then this limiting value is taken to be the sum of the series.

## Example

Let $\mathrm{a}_{\mathrm{n}}=0.5^{\mathrm{n}}$. Consider the infinite series
$\sum_{n=0}^{\infty} a_{n}$ That is, $1+(1 / 2)+(1 / 4)+(1 / 8)+(1 / 16)+\ldots$
In our formula for geometric progressions, we have $\mathrm{a}=1$, and $\mathrm{r}=0.5$. Since $|r|<1$, we can say that the infinite sum is equal to $1 /(1-0.5)=2$.

Now consider the partial sums. We have $S_{0}=1, S_{1}=1+1 / 2=1.5, S_{2}=1+1 / 2$ $+1 / 4=1.75, S_{3}=1+1 / 2+1 / 4+1 / 8=1.875$, etc.

It is intuitively easy to see (and can be proven), that this series gets closer and closer to 2 as $n$ increases (though never quite reaching 2 ). Hence, we say that $\sum_{n=0}^{\infty} a_{n}=2$ as an infinite sum.

On the other hand, if we had $\mathrm{r}=2$, so that our infinite sum was
$1+2+4+8+16+\ldots$, so that the partial sums went $1,3,7,15,31$, etc, then these partial sums are clearly not converging to any finite total, but are just increasing towards infinity. Hence this infinite series diverges, and we cannot give a value to the infinite sum.

