Production Functions

Productions functions are an important component of applied economic analysis in a number of areas:

Macro level: combined with MP theory to explain prices of factors of production and the extent to which utilised. Important for growth/distribution

Micro level: used to investigate substitutability between factors returns to scale

Both: used to consider what proportion of growth is the result of increase in factor inputs, returns to scale and technical progress

Have been object of considerable controversy -Cambridge capital controversy - but remain an important cmponent of applied economics and with the resurgence of growth theory in the form of new growth theory are increasingly important.

Consider a bit more formally:

$$Q = Q(K, L)$$

Defines the maximum output Q given inputs of capital K

and labour L. Note these are flow variables and we assume variable/divisible and continuosly substitutable. Technical questions of how to get from K and$L$ to the best Q is assumed solved, but substition means can get a given$Q$ from a number of combinations and so need to consider the minumum cost combination

$$\frac{\partial Q}{\partial L} = MP_L$$
$$\frac{\partial Q}{\partial K} = MP_K$$

which are positive but diminishing marginal productivity: diminishing returns to a factor

$$Q(K, L) = (\lambda K, \lambda L) = \lambda^n Q(k, L)$$

homogeneous of degree n, if $n < 1$ decreasing returns, $n = 1$ constant returns and $n > 1$ increasing returns.

Questions of what determines the proportions by which the factors are combined is an economic one.

At micro level use model of firm behaviour maximising profits π

$$\pi = pQ - mK - wL$$
$$\text{st } Q = Q(K, L)$$

Assuming perfect competition in production and factor markets the factor prices p, m, w are given and the Lagrangian

$$L = pQ - mK - wL - \lambda[Q - Q(K,L)]$$

An alternative economic model is to assume Q is predetermined and minimise costs subject to the level of output.

$$\min C = mK + wL$$

$$\text{st } Q^0 = Q(K,L)$$

$$L = mK + wL - \lambda[Q^0 - Q(K,L)]$$

Both the profit maximisisng and cost minimising models

imply that factors are combined so as to equate the MRS with the ratio of factor prices.

$$MRS = \frac{\partial Q}{\partial L} / \frac{\partial Q}{\partial K} = \frac{w}{m}$$

Even if the assumption of perfect competition is dropped profit maximising still implies this.

MRS measures the extent it is possible to substitute one factor for another in the production of a given output, but its size will depend on the units of measurement of L and K. For this reason the elasticity of substitution is used

$$a = \frac{d(K/L)}{K/L} / \frac{dMRS}{MRS}$$

The higher *a* the more the substituion possibilities.

Cobb-Douglas Production Function

In applied work the most commonly used form of the prodcution function is the Cobb Douglas. This resulted from Douglas observing that the share of national output going to labour was approximatley constsnt over time

$$wL = KpQ$$

An underlying production function that would give rise to this observation is of the form

$$Q = AK^{J}L^{K}$$

This has a number of convenitnt proerties J, K are the elasticities of output and A can be considered an efficiency parameter

$$\frac{/Q}{/K} = JAK^{J?1}L^{K} = J\frac{Q}{K}$$
$$\frac{/Q}{/L} = JAK^{J}L^{K?1} = K\frac{Q}{L}$$

Assuming the firm is a price taker and profit maximiser then

$$\frac{/Q}{/K} = J\frac{Q}{K} = \frac{m}{p}$$
$$\frac{/Q}{/L} = K\frac{Q}{L} = \frac{w}{p}$$

which can be written as:

$$J = \frac{mK}{pQ}$$
$$K = \frac{wL}{pQ}$$

which is the regularity Doulas observed. So if the MP conditions hold then J and K in the C-D are equal to the respective shares of capital and labour in the share of national output (also need constant returns to scale). As before this result can be derived from a cost minimising approach and for both models the optimising conditions imply:

$$\frac{K}{L} = \left(\frac{J}{K}\right)\left(\frac{w}{m}\right)$$

For any given factor price ratio the greater is J/K the greater is K/L. So size of J relative to K determines capital intensity of the production process represented by the C-D

C-D is also homogeneous of degree $J + K$

$$Q(VK, VL) = A(VK)^J(VL)^K = V^{J+K}AK^JL = V^{J+K}q(K, L)$$

$J + K > 1$ increasing returns to scale

$J + K = 1$ constant returns to scale

$J + K < 1$ decreasing returns to scale

NB the returns to scale property is the same at all levels of output and the C-D implies a constant elasticity of substitution that is equal to one

$$a = \frac{d(K/L)}{K/L} / \frac{d(w/m)}{w/m}$$

There is a problem that for the first order solutions to be unique decreasing return to scale are required, but is unlikley and if it doesnt hold vcan get some strange results. The answer is to relax the assumptions of perfect competition, where prices are exogenously given, but this make prices endogenous and suggets product demand and factor supply equations should be added to the system.

Estimating Output Elasticities

Consider:

$$Y = AK^JL^K$$

to operationalise this we normally take logs and estimate using OLS

$$\log Y = \log A + J \log K + K \log L$$

using lower case for logs we can rewrite as

$$y = J_0 + J_1 k + Kl$$

The coefficients are the elasticities

But need to consider whether the OLS assumptions hold A particular problem is likely to be the dynamics of such a model

Note that we can also use the framework to look at factor demands and productivity.

assuming constant returns to scale $J_1 + K = 1$ so

$$y = J_0 + J_1 k + \text{Ý}1 \text{ ? } J_1\text{Þ}l$$

$$y = J_0 + J_1 k + l \text{ ? } J_1 l$$

so

$$y \text{ ? } l = J_0 + J_1 k \text{ ? } J_1 l$$

$$y \text{ ? } l = J_0 + J_1\text{Ý}k \text{ ? } l\text{Þ}$$

this is sometimes used to overcome problems of multicollinearity. If we dont assume constant returns to scale then:

$$y \text{ ? } l = J_0 + J_1 k + Kl \text{ ? } l$$

$$y \text{ ? } l = J_0 + J_1 k + \text{Ý}K \text{ ? } 1\text{Þ}l$$

which can be reparametaried fro estimation as

$$y \text{ ? } l = J_0 + J_1 k + J_2 l$$

CES Production Function

The Cobb Douglas is sometimes considered a bit restrictive and there is a more general, though also more complex production function is one with constant but not necessarily unity elasticity of substitution, the CES production function.

Arrow, Chenery,Minhas and Solow (1961) estimated cross section equations

$$\frac{Q}{L} = \frac{1}{K}\left(\frac{w}{p}\right)^{e}$$

now a C-D with profit maximising and perfect competition would imply $e = 1$, but they consistently found it to be less, implying that the underlyingg function must be:

$$Q = L\text{ß}NK^{?S} + \text{Ý}1 ? N\text{þ}L^{?S}\text{à}^{?1/S}$$

where L is efficiency parameter similar to A in the C-D. This means

$$\frac{NQ}{NK} = \frac{NL}{K^{1+S}}\text{ß}NK^{?S} + \text{Ý}1 ? N\text{þ}L^{?S}\text{à}^{?\frac{\text{Ý}1+S\text{þ}}{S}} = \frac{N}{L^{S}}\left(\frac{S}{K}\right)^{1+S}$$

$$\frac{NQ}{NK} = \frac{\text{Ý}1 ? N\text{þ}L}{K^{1+S}}\text{ß}NK^{?S} + \text{Ý}1 ? N\text{þ}L^{?S}\text{à}^{?\frac{\text{Ý}1+S\text{þ}}{S}} = \frac{\text{Ý}1 ? N\text{þ}}{L^{S}}\left(\frac{S}{K}\right)^{1+S}$$

Marginal producivity equations under profit maximaisation and perfect competition are:

$$\frac{N}{L^{S}}\left(\frac{S}{K}\right)^{1+S} = \frac{m}{p}$$

$$\frac{\text{Ý}1 ? N\text{þ}}{L^{S}}\left(\frac{S}{K}\right)^{1+S} = \frac{w}{p}$$

which leads to the SMAC estimating equation:

$$\frac{Q}{L} = \frac{1}{K}\left(\frac{w}{p}\right)^{e} \text{ with } e = \frac{1}{1+S} \text{ and } \frac{1}{K} = \left(\frac{L^{S}}{1 ? N}\right)$$

MRS is

$$\left(\frac{1 ? N}{N}\right)\left(\frac{K}{L}\right)^{1+S}$$

and elasticity of substitution is:

$$a = \frac{1}{1+S}$$

S is known as substitution parameter as $S = \frac{1}{a} ? 1$

t when $S = K$ then $a = 0$ and substitution is impossible
t when $S = ?1$ then $a = K$ and isoquants are straight lines
t when $S = 0$ then $a = 1$ and we have the C-D production function

Can also write

$$\frac{wL}{mK} = \left(\frac{1 ? N}{N}\right)\left(\frac{K}{L}\right)^{S}$$

So for a given K/L and S, N will determine the shares of capiatl and labour. It is the ”distribution parameter”. For the C-D the ratio of factor shares was constant.

Note that the CES above implies constant returns to scale, but we can generalise to

$$Q = L\text{ß}NK^{?S} + \text{Ý}1 ? N\text{Þ}L^{?S}\text{à}^{?v/S}$$

v is then the returns to scale factor:

t $v > 1$ increasing returns to scale
t $v = 1$ constant returns to scale

- $v > 1$ decreasing returns to scale

Applying the models

Serious conceptual problems with aggregate production functions:

- Capital controversy: theoretical inconsistencies in N-C production function and distribution.
- Individual firms unlikely to have the same production function. C-D is in logs so means should be geometric.
- Production function is only one equation of a simultaneous system: marginal productivity conditions, need to be aggregated as well.
- Existence of external economies of scale: whole > sum of parts
- Firms/industries can have widely different outputs and techniques of production: J and K unlikely to be distributed independently of K and L. High values of K are likely to have high J and high L high K. These may change over time. Increases in output will be greater if increase labour in a labour intensive industry, but where it goes depends on factor prices and these may vary in non-competitive industries. Cannot be a purely technical relationship.

Serious practical problems

- Aggregating labour inputs: use person hours, but how to treat skills/quality . In practice approximate by aggreagate monetary value of inputs, deflated by a labour input price index, or use unweighted measures of flows (total person hours) or stock (total no. employees). Also use of gross output and value added measures.
- Aggregating capital inputs causes greatest problems: Often use value of the capital stock (replacement cost in some base year gross or net of depreciation) in practice, but there are many problems (variations in quality, scrapping, using stock rather than flow). Sometimes assume all revenues go to labour ad capital (no profits) so $PQ = wL + mK$ but this can lead to serious problems of interpretation. Problems in getting a price of capital when needed.