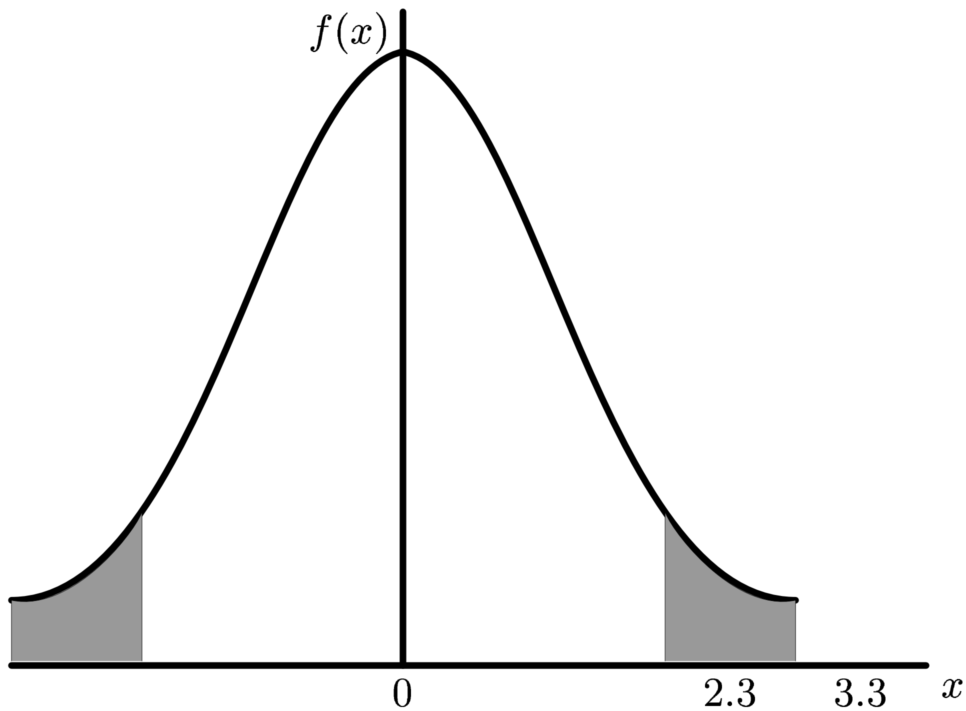
The principles of specification testing

- Recall that our estimates of $\hat{\alpha}$ and $\hat{\beta}$ are random variables. That is, they are particular realisations of values from a distribution. For $\hat{\beta}$, this distribution is $N(\beta, \sigma^2 / \sum(x_i - \bar{x})^2)$. This means that though our estimate may be positive, it may in fact not be *statistically* different from zero.
- Suppose, for example, our estimate of $\hat{\beta}$ was 2.3. Consider the following distribution which assumes that the true value of β is 0.



- This has a mean of zero, but there is still positive probability that a value of 2.3 will occur. There is less probability that a realisation of the value 4.3 will occur and even less that a value of 213 will occur. We want to be sure that our value (2.3) is sufficiently far into the tails of this zero-mean distribution that we can be sure that the value did not come from this distribution. In other words, we need to be sure that this does not look like the distribution from which our value of $\hat{\beta}$ arose.

- To do this we need to ensure that our value is sufficiently in the tails of the zero-centred distribution that we can be reasonably sure that 0 is not the true value. It is customary to choose the tails so that they constitute 5% of the entire probability space. This would typically mean 2.5% area in the lower tail and 2.5% area in the higher tail for a *two-tailed* test. For the normal distribution these *critical values* are $+/-1.96$. This would be termed a two-tailed test at a 5% significance level.

- Consider the null (or maintained) hypothesis:

$$H_0 : \hat{\beta} = \beta_R$$

and the alternative hypothesis:

$$H_1 : \hat{\beta} \neq \beta_R$$

- Now consider the random variable:

$$\begin{aligned} Z &= \frac{(\hat{\beta} - \beta_R)}{\sqrt{\mathrm{V}(\hat{\beta})}} \\ &= \frac{(\hat{\beta} - \beta_R)}{s.e(\hat{\beta})} \end{aligned}$$

- Under our classical assumptions, $\hat{\beta}$ is normally distributed as $N(\beta, \sigma^2 / \sum (x_i - \bar{x})^2)$. Thus the above variable is a standard normal variable.

- However, we need to estimate the unknown quantity, σ^2. We can do this with the unbiased formula

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} .$$

- Now consider the distribution of $\hat{u}_i$. We know that this is $N(0, \sigma^2)$. Thus we can standardise it to $N(0, 1)$ by dividing by σ. Then if we sum across all time periods we get a $\chi^2(n-2)$ variable (recall that only $n-2$ of these variables are independent). i.e.

$$\sum \frac{\hat{u}}{\sigma} \sim \chi^2(n-2)$$

- Now divide the standard normal variable Z from above by the square root of this $\chi^2(n-2)$ variable divided by its degrees of freedom to form a t-distributed variable.

$$t = \frac{(\hat{\beta} - 0)/\sqrt{\sigma^2/\sum(x_i - \bar{x})^2}}{\sqrt{\sum \hat{u}_i^2/\sigma^2(n-2)}}$$

- This leaves:

$$t = \frac{\hat{\beta} - \beta}{\sqrt{\hat{\sigma}^2/\sum(x_i - \bar{x})^2}}$$

or, alternatively

$$t = \frac{\hat{\beta} - \beta}{\widehat{s.e.}(\hat{\beta})}$$